International Journal of Mathematical Archive-8(1), 2017, 190-193
Available online through www.ijma.info ISSN 2229 – 5046

APPROXIMATIONS OF FIXED POINT MAPPINGS IN BANACH SPACES

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(Received On: 17-12-16; Revised & Accepted On: 17-01-17)

ABSTRACT

In this paper we prove some results concerning approximations of Fixed points of pair of mappings in Banach ***spaces.***

1 INTRODUCTION

Let X be a Banach space with norm ||.|| and C be closed, convex and bounded subset of X. In [9] Dotson introduced the concept of quasi-nonexpansive mappings which is as follows:

Definition 1.1: A self-mapping T of C is said to be quasi-nonexpansive provided T has a fixed point in C and if

$$\|T(x) - p\| \leq \|x - p\| \tag{1.1}$$

is true for all $x \in C$

In the following we consider a new class of contraction mappings.

Definition 1.2: A self-mapping T of C with nonempty fixed point set F(T) in C is said to be a F-contraction if $p \in F(T)$, then for some q, $0 \leq q < 1$;

$$\|T(x) - p\| \leq q\,\|x - p\| \tag{1.2}$$

is true for all $x \in C$

It is clear that every F-contraction mapping is contained in the class of quasi-nonexpansive mappings.

In a uniformly convex Banach space, Senter and Dotson [8] have given condition under which Mann type iterates of quasi-nonexpansive mappings converge to fixed point of the mapping. Here we consider a pair of mappings in Banach space satisfying quasi-nonexpansive type conditions of Ciric [1] and using an approach similar to Senter and Dotson [8], we prove some results concerning the approximation of fixed points of such pairs of mappings. For this we need the following conditions given by Senter and Dotson [8].

Condition 1: Let $T: C \to C$ be with a nonempty fixed point set F(T) in C such that there is a non-decreasing function $f: [0;\infty) \to [0;\infty)$ with $f(0) = 0$; $f(t) > o$ for $t \in (0; \infty)$, satisfying

$$\|x - Tx\| > f(d(x; F(T))) \tag{1.3}$$

For $x \in C$ where $d(x, F(T)) = \inf\{\|x-p\|: p \in F(T)\}$.

Condition 2: Let $T: C \to C$ be with a nonempty fixed point set F(T) in C such that there is a real number $\alpha > 0$ for which

$$\|x - Tx\| > \alpha(d(x; F(T))) \tag{1.4}$$

holds for all $x \in C$

It can be easily seen that the mapping satisfying condition 2 also satisfies condition 1.

We need the following Theorem [10]

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Theorem 1.1: Let C be a nonempty, closed convex and bounded subset of normed linear space X and let T_1, T_2:C→ C be two mappings satisfying conditions:

i) $\|T_1x-T_2y\| \leq a(\|x-T_1x\|+\|y-T_2y\|)+b \max(\|x-y\|, \|x-T_1x\|,\|y-T_2y\|, 1/2(\|x-T_1y\|+\|y-T_1x\|)\}$ (1.5)
for x, y Є C, where a and b are nonnegative real numbers satisfying a + b < 1.
and
ii) for some x_0 Є C the sequence $\{x_n\}$ where

$$x_{2n+1} = (1 - t)x_{2n} + t T_1x_{2n},$$
$$x_{2(n+1)} = (1 - t)x_{2n+1} + t T_2x_{2n+1},$$ n = 0; 1, 2,… and t Є (0, 1) converges to a point u.

Then u is a common fixed point of T_1 and T_2.

2. MAIN RESULT

Theorem 2.1: Let X be a uniformly convex Banach space. Let C be a nonempty, closed, convex and bounded subset of X. If T_1 and T_2 are two mappings on C into itself satisfying condition (1.5) with a+ b < 1 ⁄ 2. Then for arbitrary x_1, x'_1 Є C, $M(x_1, t_n, T_1)$ and $M(x'_1, t_n, T_2)$ converge to a unique common fixed point of T_1 and T_2.

Proof: By Theorem 1.1, T_1 and T_2 have a unique fixed point, say p Є C. If u and w are fixed points of T_1 and T_2 respectively, then v = w = p. For suppose that v ≠ w, then by condition (1.5), we get

$$\begin{aligned}\|v - w\| &= \|T_1v - T_2w\| \leq a (\|v - T_1v\| + \|w - T_2w\|) + b \max \{\|v - w\|, \|v - T1v\|, \|w - T_2w\|, \\ &\qquad 1/2 (\|v - T_1w\| + \|w - T_1v\|)\} \\ &= a (\|v - v\| + \|w - w\|) + b \max \{\|v - w\|, \|v - v\|, \|w - w\|, 1/2 (\|v - w\| + \|w - v\|\} \\ &= b \max (0; \|v - w\|) \\ &= b \|v - w\|\end{aligned}$$

which is contradiction since b < 1. Hence v = w is common fixed point of T_1 and T_2. By Uniqueness of the common fixed point p of T_1 and T_2, we have v = w = p.

Therefore, to prove the Theorem, we only show that the sequences $M(x_1, t_n, T_1)$, $M(x'_1, t_n, T_2)$ Converge to the fixed points of T_1 and T_2 respectively. Clearly, each of T_1 and T_2 is quasi-nonexpansive on C.

For, let x Є C, then

$$\begin{aligned}\| T_1x - p\| = \| T_1x - T_2p\| &\leq a(\|x - T_1x\| + \|p - T_1p\|)+ b \max\{\|x - p\|, \|x - T_1x\|, \|p - T_2p\|, \\ &\qquad 1/2 (\|x - T_2p\| + \|p - T_1x\|)\} \\ &= a\|x - T_1x\| + b \max\{\|x - p\|, \|x - T_1x\|,1/2(\|x - p\|+ \|x - T_1x\|)\} \qquad (2.1)\end{aligned}$$

Now there are three cases:

Case-1: If $\max \{\|x-p\|; \|x- T_1x\|; 1/2 (\|x-p\|; \|p- T_1x\|)\} = \|x-p\|$

$$\begin{aligned}\|T_1x-p\| &\leq a \|x- T_1x\| +b\|x-p\| \\ &\leq a\|x - p\| + a\|T_1x - p\| + b\|x - p\|\end{aligned}$$

i:e, $$\|Tx_1 - p\| \leq \Gamma_1\|x - p\| \qquad (2.2)$$

where $\Gamma_1= (a+b) / (1-a) < 1$ Since, a + b < 1 ⁄ 2 .

Case-2: If $\max \{\|x-p\|; \|x-T_1 x\|; 1/2 (\|x-p\|; \|p-T_1x\|)\} = \|x-T_1x\|$

$$\begin{aligned}\|T_1x-p\| &\leq a\|x-T_1x\|+b\|x-T_1x\| \\ &= (a + b) \|x - T_1x\| \\ &\leq (a + b) \|x - p\| + (a + b)\|T_1x - p\|\end{aligned}$$

$$\|T_1x - p\| \leq \Gamma_2\|x - p\| \qquad (2.3)$$

where $\Gamma_{2=} (a+b) / 1-(a+b) < 1$

Since, a + b < 1.

Case-3: If $\max \{\|x - p\|, \|x - T_1x\|, 1/2 (\|x - p\|+ \|p - T_1x\|)\} = 1/2 (\|x - p\|+ \|p - T_1x\|)$

$$\begin{aligned}\|T_1x - p\| &\leq a\|x - T_1x\| + b\, 1/2 (\|x - p\| + \|p - T_1x\|) \\ &\leq a\|x-p\|+a\|T_1x-p \|+b/2\|x-p \|+ b/2 \| p-T_1x\| \\ &= (a+b/2) \| x-p \| + (a+b/2)\|T_1x-p \| \\ &\leq (a+b/2) / (1-(a+b/2)) \|x-p \| \\ \|T_1x - p\| &\leq \Gamma_3 \| x-p\|\end{aligned}$$

Where $\Gamma_3 = (a+b/2) / (1-(a+b/2)) < 1$ since $a + b < 1/2$.

Let $\Gamma = \max(\Gamma_1, \Gamma_2, \Gamma_3)$ then from (2.2), (2.3) and (2.4) we obtain

$$\| T_1x-p\| \le \alpha \|x - p\| \quad \text{for all } x \in C.$$

This shows that T_1 is F-contraction and hence T_1 is quasi-nonexpansive on C. Similarly it can be shown on same line that T_2 is quasi-non expansive on C.

Next we show that T_1 and T_2 satisfy condition 2 on C.

For this we see again three cases

Case-1: If $\max\{\|x-p\|, \|x-T_1x\|, 1/2(\|x-p\|+\|p-T_1x\|)\} = \|x-p\|$. Then from (2.1) we obtain

$$\|T_1x-p\| \le a\|x-T_1x\|+b\|x-p\|$$

$$\|x-p\| - \|x-T_1x\| \le \|T_1x-p\|$$
$$\le a\|x-T_1x\| + b\|x-p\|$$

$$(1-b)\|x-p\| \le (1+a)\|x-T_1x\|$$

$$\|x-T_1x\| \ge (1-b)/(1+a)\|x-p\|$$

$$\|x-T_1x\| \ge \alpha_1\|x-p\|$$

For all $x \in C$ where $\alpha_1 = (1-a)/(1+a) > 0$

Case-2: If $\max\{\|x-p\|, \|x-T_1x\|, 1/2(\|x-p\|+\|p-T_1x\|)\} = \|x-T_1x\|$

By (2.1) we obtain

$$\|T_1x-p\| \le a\|x-T_1x\| + b\|x-T_1x\|$$
$$= (a+b)\|T_1x-p\|$$
$$\|x-p\| - \|x-T_1x\| \le \|x-T_1x\|$$

$$\|x-p\| - \|x-T_1x\| \le (a+b)\|x-T_1x\|$$

$$\|x-p\| \le (1+a+b)\|x-T_1x\|$$

$$\|x-T_1x\| \ge 1/(1+a+b)\|x-p\|$$

$$\|x-T_1x\| \ge \alpha_2\|x-p\|$$

For all $x \in C$ where $\alpha_2 = 1/(1+a+b) > 0$.

Case-3: If $\max\{\|x - p\|, \|x - T_1x\|, 1/2(\|x - p\|+\|p - T_1x\|)\} = 1/2(\|x - p\|+\|p - T_1x\|)$

$$\|T_1x - p\| \le a\|x - T_1x\| + b\,1/2(\|x - p\| + \|p - T_1x\|)$$

$$(1-b/2)\|T_1x-p\| \le a\|x-T_1x\|+b/2\|x-p\|$$

$$(1-b/2)\{\|x-p\| - \|x-T_1x\|\} \le a\|x-T_1x\|+b/2\|x-p\|$$

$$(a+(2-b)/2)\|x-T_1x\| \ge ((2-b)/(2-b/2)\|x-p\|$$

$$\|x-T_1x\| \ge (2-2b)/(2a-b+2)\|x-p\|$$

$$\|x-T_1x\| \ge \alpha_3\|x-p\| \text{ for all } x \in C \text{ where } \alpha_3 = (2-2b)/(2a-b+2) > 0.$$

Let $\alpha = \min(\alpha_1, \alpha_2, \alpha_3)$ then

$$\|x-T_1x\| \ge \alpha\|x-p\|,\ \alpha > 0, \text{ for all } x \in C.$$

Thus T_1 satisfies condition 2 and hence condition 1 on C. Repeating the similar arguments, it can be shown that T_2 satisfies the condition 2 and hence 1 on C. Thus by theorem 3[8], for arbitrary x_1, $x'_1 \in C$, the sequences $M(x_1, t_n, T_1)$ and $M(x'_1, t_n, T_2)$ converge to a fixed point T_1 and T_2 respectively. Consequently the sequences $M(x_1, t_n, T_1)$ and $M(x'_1, t_n, T_2)$ converge to a unique common fixed point of T_1 and T_2. This complete the proof.

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We note that by taking $T_1 = T_2 = T$ in the above Theorem we get the following result as a corollary.

Corollary 2.1: Let C be non-empty, closed, convex and bounded subset of uniformly convex Banach Space X. If T is a mapping on C into itself satisfying the condition (1.5), with a +b < 2, then for arbitrary $x_1 \in C$, the sequence $M(x_1, t_n, T)$ converges to a unique fixed point of T.

Theorem 2.2: Let C be a nonempty, closed, convex and bounded subset of uniformly convex Banach spaces X and let $T_1\ T_2$: $C \to C$ be two mappings such that there exists r and t satisfying condition

$$\|T^r{}_1x-T^s{}_2y\| \leq a(\|x- T^r{}_1x\|+\|y- T^s{}_2y\|)+b\max\{\|x-y\|, \|x- T^r{}_1x\|, \|y- T^s{}_2y\|, ½(\| x- T^s{}_2y\|+\|y- T^r{}_1x\|)\} \tag{2.5}$$

For every x, y $\in$ C and a+b<1/2. Then for arbitrary x_1, $x'_1 \in C$ the sequences $M(x_1, t_n, T^r{}_1)$ and $M(x'_1, t_n, T^s{}_2)$ converge to a unique common fixed point of T_1 and T_2.

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Source of support: Nil, Conflict of interest: None Declared.

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