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WEAK CS-MODULES AND FINITE DIRECT SUM OF INJECTIVE MODULES

NIDHI MISHRA, U. K. KHEDLEKAR*

Department of Mathematics and Statistics,
Dr. Harisingh Gour Vishwavidyalaya, (A Central University), Sagar (M.P.)-India.

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ABSTRACT

In this paper, we investigate a generalization of CS-modules (or extending modules) called weak CS-modules, study the nature of their interaction with other modules and find results to show their relationship with CS-modules, quasi-continuous modules and other modules. We emphasize the study of finite direct sum of modules, their interaction with other modules namely, quasi-continuous modules, CS-modules and weak CS-modules.

Keywords: *Weak CS-modules, CS modules, quasi-continuous module.*

1. INTRODUCTION

A module M is called CS (or extending) if every submodule of M is essential in its direct summand. Alternatively, a module M is called CS (or extending) if every compliment of M is a direct summand. It is well known that every CS module is CESS-module and every CESS-module is *weak* CS-modules [1]. Some properties of weak CS-modules behave like the of CS modules. It was drafted in [2], that a finite direct sum of relatively injective weak CS-modules is weak CS. In this paper, we discuss the properties of weak CS-modules and study the relationship between weak CS-modules, CS-modules, uniform modules, quasi-continuous (QC) modules and other modules.

2. PRELIMINARIES

Throughout this paper, R will denote a ring with identity and M a unitary right R- module. A right R-module M is said to be *indecomposable* if it is nonzero and cannot be expressed as a direct sum of two nonzero R-submodules of M. We *consider the following conditions for a module M:*

(C_1) Every submodule of M is essential in a direct summand of M.
(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand of M.
(C_3) If A, B are direct summand of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

A module M is *CS* if it satisfies condition (**C_1**). A module M is called a *CESS-module* if every complement in M with essential socle is a direct summand of M.

Definition 2.1: A module M is called *weak CS-module* if every semisimple submodule of M is essential in a direct summand of M. Semisimple modules, (Quasi-) injective modules, (Quasi-) continuous modules are all examples of weak CS-modules.

Definition 2.2: A module M is called *weak quasi-continuous* if M is a weak CS-module and satisfies the condition (C_3). And a module M is called *weak continuous* if M is a weak CS-module and satisfies the condition (C_2).

Definition 2.3: A module is called a *uniform* module if the intersection of any two non-zero submodules is nonzero. Equivalently, M is uniform if every nonzero submodule of M is essential in M.

Lemma 2.1: ([1], Lemma 1.1) *Every CS-module is weak CS-module*.

Remark: The converse of the above lemma *may not hold true in general.* Consider the following example.

Corresponding Author: U. K. Khedlekar*
Department of Mathematics and Statistics,
Dr. Harisingh Gour Vishwavidyalaya, (A Central University), Sagar (M.P.)-India.

Example 2.1: The Z-module M = Z/pZ $\oplus$ Q , where p is prime, is a weak CS-module but not a CS-module.
The Z-module Z/pZ is simple and hence CS and Q is an injective Z-module. Clearly, M is a weak CS-module but not a CS-module. As we see that M satisfies the condition (C_2) but not (C_1). Similarly, we can say that M is a weak continuous module but not a continuous module.

Lemma 2.2: ([2] Corollary 1.5; [5])
2.2.1 Any direct summand of weak CS-module is weak CS.
2.2.2 Any direct summand of weak QC-module is weak QC.

Lemma 2.3: ([2], Theorem 1.9)
If $M = M_1 \oplus \ldots\ldots \oplus M_n$ *is a finite direct sum of weak CS-modules* M_i *, where for each i,* M_i *is* M_j*-injective,* $j \neq i$*, then M is a weak CS-module.*

We shall now proceed to the main findings of this paper.

Theorem 3.1: *Let* $M = M_1 \oplus \ldots\ldots \oplus M_n$ *be a finite direct sum of modules. If M is a quasi-continuous module, then M is a weak CS-module*.

Proof: If $M = M_1 \oplus \ldots\ldots \oplus M_n$ is quasi-continuous module, then each M_i is quasi-continuous and M_j-injective for all j>i, by ([5], Corollary 2.14)

It is known that any quasi-continuous module is CS, which implies that M_i is CS and by Lemma 2.1, one can conclude that each M_i is weak CS-module.

Then, for $M = \oplus_{i=1,..,n} M_i$, where each M_i is M_j-injective, j > i, M is weak CS-modules by Lemma 2.3.

Corollary 3.2: *A finite direct sum of relatively injective QC module is weak CS-module.*

Proof: Let $M = M_1 \oplus \ldots\ldots \oplus M_n$, where each M_i is relatively injective quasi-continuous module. Then, according to [5, Corollary 2.14], M is quasi-continuous and hence a CS-module.

Now as we know, by lemma 2.1, that any CS-module is weak CS-module, we finally conclude that M is a weak CS-module.

Example 3.1: ([4], Example 3.2.5)

Consider a Z-module $M = Z_2 \oplus Z_8$, where Z_2 and Z_8 are weak CS but not relatively injective and M being a weak CS-module.

Proof: An Z_2 is simple, Z_8 is uniform, clearly Z_2 and Z_8 are weak CS-modules. But Z_2 is not Z_8 injective.

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