PROBABILITY MEASURE ON SMOOTH MANIFOLDS AND RANDOMNESS

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ABSTRACT

***R**andomness on smooth manifolds is now being extensively studied for various reasons. The purpose of this paper is to give a brief account of such a process on smooth manifolds which admits a probability measure. The exposition would also gives some new developments that have come to fore as a result of randomness entering the realm of analysis and geometry.*

Keywords: Probability measure, event, smooth manifold and randomness.

1. INTRODUCTION

Axiomatic development of mathematical theory of probability was initiated by A.N.Kolmogrov. His 1930's, initiation to firmly lay the foundation of probability theory based on measure theoretic notions for a probability space gave enormous scope to many analysts, scientists and engineers to take advantage of this treatment to the problems that they encountered. Randomness as such was not properly understood, before an axiomatic approach to the theory of probability was provided. Various processes involving randomness (called also as, Stochastic processes) required the knowledge of a probability space as envisioned by A.N. Kolmogrov and his successes, [1-3]. In this paper we are introducing the notion of a probability space as a smooth manifold and try to figure out randomness on them.

2. PROBABILITY MEASURE SPACE AND RANDOM VARIABLES

Probability measure space and random variables is a triple $(\Omega, \mathcal{F}, P)$ where Ω is a sample space and its points are called sample points. $\mathcal{F}$ is the $\sigma - field$ whose members are the events (here $\mathcal{F}$ is a event space) and P is called probability measure defined on the measurable space $(\Omega, \mathcal{F})$. If an event A is given by A= {w ∈ Ω/R(w)} for some property R(.) then we may write R(A) P(A) or p(A) An event is called sure event if P(A)=1 i.e something will occur. This in $\mathcal{F}$ are the corresponding members, $\Omega, Q \in \mathcal{F}$, respectively.

In a probability space like the one defined above, the notion of a random variable is very important and it is through random variable random processes are modeled in a probability space. To this end let $(\mathbb{E}, \zeta)$ be another measurable space.

Suppose we pick a point ω from Ω (by assuming a throught experiment of flipping a fair coin) this gives us a map X:$\Omega \longrightarrow E$ and will enable us to form sets of the type {ω/ x(ω)∈B}, where B⊂ E.

In other words, element of $\mathcal{F}$ are the collection of subsets and denoted by $\mathbb{E}$. In it now boils down and a measurable space $(\mathbb{E}, \zeta)$ is formed. The map X:$(\Omega, \mathcal{F}) \longrightarrow (\mathbb{E}, \zeta)$ has to be measurable.

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Such a map X will be called a $\mathbb{E}$-valued random variable. This formalism gives us the required mathematical frame work for probability related problems. Kolmogrov, Markov, Borel, were early point of 20th century. One would also refer the works of [4-9].

Remark 2.1: The measurable space (E, ζ) will be have a probability measure with the probability measure with the probability p induced by it on (E, ζ). Since X: $(\Omega, \mathcal{F}, P) \longrightarrow (E, \zeta)$ is measurable the image of μ of P. The probability measure is (E, ζ) and is called law of X.

A random variable (or map) X: $(\Omega, \mathcal{F}, P) \longrightarrow (E, \zeta)$ gives rise to a probability measure space (E, ζ, μ), Where μ is the image of p under X.

Proposition 2.2: The events $\{\omega / x(\omega) \in A\}$ forms a sub σ- field of $\mathcal{F}$.

Proof: Since each $\{\omega / x(\omega) \in A\}$ is a subset A of (E, ζ) and ζ is a σ- field. Whose member are sub family of $\mathcal{F}$ measurable so from a sub σ- field of $\mathcal{F}$. The σ- field in the proposition is called σ- field generated by X.

3. SMOOTH MANIFOLDS AS PROBABILITY SPACES

Generally, we have $\mathbb{R}^N$ as the sample space, $\sigma - Borel$ field with $\mathbb{R}^N$ becomes a measurable space. If $\mathcal{B}(\mathbb{R}^N)$ denotes the $\sigma -$ field. The pair $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ is the measurable space.

To this end we consider the smooth manifold M of dimension N, if $\mathcal{F}$ is the $\sigma -$ field associated with M, the measurable space (M, $\mathcal{F}$) admitting a probability measure with become a measurable space on this space We will define random variable in term see that it generates a sub-$\sigma -$ field of $\mathcal{F}$, the measurable space is $(\mathbb{E}, \zeta)$ associated with the probability space $(M, \mathcal{F}, P)$ is the image of P.

Proposition 3.1: If $\{X_\alpha : \alpha \in \Lambda\}$ is a family of random variables in $(M, \mathcal{F}, P)$. Then, each X_α generates a sub $-\sigma -$field on the event space $(E_\alpha, \zeta_\alpha), \alpha \in \Lambda$.

We shall obtain a sub $\sigma -$ field associated with the probability space $(M, \mathcal{F}, P)$, where M is a smooth N - dimensional manifold, $\mathcal{F}$ a $\sigma -$ field on M and P the probability measure. Since, M admits a smooth differential structure, arising from the maximal atlas, which in terms induced topology on M. From this back ground it is not difficult to come of with a $\sigma -$ field on M. Let $\mathcal{F}$ be the $\sigma -$ field. Then (M, $\mathcal{F}$) is a measurable space. If P is the probability measure, then $(M, \mathcal{F}, P)$ becomes the probability measure space.

Let T_x M be the tangent space of M at x and it is isomorphic to $\mathbb{R}^n$, with induced vector space structure on it by the $\mathbb{R}^n$, we can define a $\sigma -$field ζ on T_x M, and make $(T_x M, \zeta)$ a sub-$\sigma -$field of $\mathcal{F}$.

For each x∈M, Let TM= $U_x T_x$ M, gives rise to smooth bundle, called the tangent bundle of M and whose dimension is twice the dimension of M.

If V: M→TM is a vector field, then to each $p \in M$, P→ X_p tangent vector is some T_pM of M in TM.

Since, we are interested in $(T_x M, \zeta)$ as an event space. We want a random variable to be a map X: (M. $\mathcal{F}, p) \rightarrow (T_x M, \zeta)$.

Since V_P is the tangent vector to M at p, and X:(M,$\mathcal{F}$,P)→($\mathbb{E}, \zeta$), is a random variable, then to each $\omega \in M, \omega \rightarrow X(\omega)$ is a point in E.

In other words, we are looking for $\{\omega / X(\omega) \in B\}$, where $B \subset E$. In other words, for a suitable collection of $\omega's$ in the Borel field of $\mathbb{E}$, the set $\{\omega / X(\omega) \in B\} \in$ sub-σ-field of $\mathbb{E}$. We have already noticed this fact (in our earlier section).

Replacing $\mathbb{E}$ by TM and regarding T_x M as an isomorphic copy of $\mathbb{R}^n$, the σ-Borel field $\mathcal{B}(\mathbb{R}^n)$ induces a σ-Borel field on TM.

Since X: M→TM is a vector field restricting the points of M to the suitable fibers that come from projection map π: $TM \rightarrow M$, $\pi^{-1}(U)$ for some chart member (U, φ) in M. To this end, we obtain that the map x→ $X(x) \in T_x$ M⊂TM. The fibers $\pi^{-1}(U)$ are in T_x M's.

This establishes our claim to construct a sub $\sigma -$field induced by the random variable.

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