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CERTAIN NEW CLASSES CONTAINING COMBINATION OF RUSCHEWEYH DERIVATIVE AND A NEW GENERALIZED MULTIPLIER DIFFERENTIAL OPERATOR

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ABSTRACT

Certain new classes containing the linear operator obtained as a linear combination of Ruscheweyh derivative and a new generalized multiplier differential operator have been considered. Sharp results concerning coefficients, distortion theorems of functions belonging to these classes are discussed.

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1. INTRODUCTION

Denote by U the open unit disc of the complex plane, $U = \{z \in C : |z| < 1\}$. Let $H(U)$ be the space of holomorphic functions in U. Let A denote the family of functions in $H(U)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

In [19], S R Swamy has introduced the following new generalized multiplier differential operator (See [17] also).

Definition 1.1: Let $m \in N_0 = N \cup \{0\}, \beta \geq 0, \gamma \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then for $f \in A$, a new generalized multiplier operator $I^m_{\alpha,\beta,\gamma}$ was defined by

$$I^0_{\alpha,\beta,\gamma} f(z) = f(z), I^1_{\alpha,\beta,\gamma} f(z) = \frac{\alpha f(z) + \beta z f'(z) + \gamma z^2 f''(z)}{\alpha + \beta}, \ldots, I^m_{\alpha,\beta,\gamma} f(z) = I_{\alpha,\beta,\gamma}(I^{m-1}_{\alpha,\beta,\gamma} f(z)).$$

Remark 1.2: Observe that for $f(z)$ given by (1.1), we have

$$I^m_{\alpha,\beta,\gamma} f(z) = z + \sum_{k=2}^{\infty} \Phi_k(\alpha,\beta,\gamma,m) a_k z^k, \tag{1.2}$$

where

$$\Phi_k(\alpha,\beta,\gamma,m) = \left(\frac{\alpha + k\beta + k(k-1)\gamma}{\alpha + \beta}\right)^m. \tag{1.3}$$

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We note that:

i) $I^m_{\alpha,\beta,0} f(z) = I^m_{\alpha,\beta} f(z)$ ([18])

ii) $I^m_{1-\beta,\beta,0} f(z) = D^m_\beta f(z),\ \beta \ge 0$ ([1]),

iii) $I^m_{l+1-\beta,\beta,0} f(z) = I^m_{l,\beta} f(z),\ \ l > -1,\ \ \beta \ge 0$ ([3] and it has been considered for $l \ge 0$) and

iv) $I^m_{1-\lambda+\mu,\lambda-\mu,\lambda\mu} f(z) = D^m_{\lambda,\mu} f(z), \lambda > (\mu/(\mu+1)), \mu \ge 0$ ([7] and they have examined for $\lambda \ge \mu \ge 0$).

Remark 1.3: i) $I^m_{1-\lambda+\mu,\lambda-\mu,\lambda\mu} f(z) = D^m_{\lambda,\mu} f(z),\ \lambda \ge \mu \ge 0,$ was also studied by Raducanu in [8]. ii) $D^m_1 f(z)$ was introduced by Salagean [10] and was considered for $m \ge 0$ by Bhoosnurmath and Swamy in [2].

Definition 1.4: ([9]) For $m \in N_0$, $f \in A$, the operator R^m is defined by $R^m : A \to A$,

$$R^0 f(z) = f(z)\,, R^1 f(z) = zf^{'}(z),\ldots,(m+1)R^{m+1} f(z) = z(R^m f(z))^{'} + mR^m f(z), z \in U.$$

Remark 1.5: If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$, then $R^m f(z) = z + \sum_{k=2}^{\infty} \Omega_k(m) a_k z^k,\ \ z \in U$, where

$$\Omega_k(m) = \frac{(m+k-1)!}{m!(k-1)!}. \tag{1.4}$$

We now state the following new operator, introduced by us in [6]:

Definition 1.6: Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \ge 0, \beta \ge 0, \gamma \ge 0, \alpha$ a real number such that $\alpha + \beta > 0.$ Denote by $RI^m_{\alpha,\beta,\gamma,\delta}$, the operator given by $RI^m_{\alpha,\beta,\gamma,\delta} : A \to A,$

$$RI^m_{\alpha,\beta,\gamma,\delta} f(z) = (1-\delta)R^m f(z) + \delta I^m_{\alpha,\beta,\gamma} f(z), z \in U.$$

Clearly i) $RI^m_{\alpha,\beta,0,\delta} = RI^m_{\alpha,\beta,\delta}$ [13], [14], [15] and [16], ii) $RI^m_{\alpha,\beta,\gamma,0} = R^m$ [10] and iii) $RI^m_{\alpha,\beta,\gamma,1} = I^m_{\alpha,\beta,\gamma}$ [19].

Remark 1.7: If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then from (1.2) and Remark 1.4, we have

$$RI^m_{\alpha,\beta,\gamma,\delta} f(z) = z + \sum_{k=2}^{\infty} \{(1-\delta)\Omega_k(m) + \delta \Phi_k(\alpha,\beta,\gamma,m)\} a_k z^k,\ \ z \in U,$$

where $\Phi_k(\alpha,\beta,\gamma,m)$ and $\Omega_k(m)$ are as defined in (1.3) and (1.4), respectively.

Motivated by a paper of Swamy [12] we now introduce new classes, shown below:

Definition 1.8: Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \ge 0, \rho \in [0,1), \sigma \in (0,1], \beta \ge 0, \gamma \ge 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $S^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$ if and only if

$$\left| \frac{\dfrac{z(RI^m_{\alpha,\beta,\gamma,\delta} f(z))^{'}}{RI^m_{\alpha,\beta,\gamma,\delta} f(z)} - 1}{\dfrac{z(RI^m_{\alpha,\beta,\gamma,\delta} f(z))^{'}}{RI^m_{\alpha,\beta,\gamma,\delta} f(z)} + 1 - 2\rho} \right| < \sigma\ , z \in U\,. \tag{1.5}$$

Definition 1.9: Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \ge 0, \rho \in [0,1), \sigma \in (0,1], \beta \ge 0, \gamma \ge 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $K^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$ if and only if

$$\left| \frac{\dfrac{[z^2(RI^m_{\alpha,\beta,\gamma,\delta} f(z))^{'}]^{'}}{(zRI^m_{\alpha,\beta,\gamma,\delta} f(z))^{'}} - 1}{\dfrac{[z^2(RI^m_{\alpha,\beta,\gamma,\delta} f(z))^{'}]^{'}}{(zRI^m_{\alpha,\beta,\gamma,\delta} f(z))^{'}} + 1 - 2\rho} \right| < \sigma\ , z \in U\,. \tag{1.6}$$

Definition 1.10: Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1), \sigma \in (0,1], \beta \geq 0, \gamma \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $C^m_{\alpha,\beta,\gamma,\delta}(\sigma, \rho)$ if and only if

$$\left| \frac{\frac{[z(RI^m_{\alpha,\beta,\gamma,\delta} f(z))']'}{(RI^m_{\alpha,\beta,\gamma,\delta} f(z))'} - 1}{\frac{[z(RI^m_{\alpha,\beta,\gamma,\delta} f(z))']'}{(RI^m_{\alpha,\beta,\gamma,\delta} f(z))'} + 1 - 2\rho} \right| < \sigma \ , z \in U . \tag{1.7}$$

Definition 1.11: Let $f \in A, m \in N_0 = N \cup \{0\}, \lambda \geq 0, \delta \geq 0, \rho \in [0,1), \sigma \in (0,1], \beta \geq 0, \gamma \geq 0, \ \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\mathrm{P}^m_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma, \rho)$ if and only if

$$\left| \frac{(1-\lambda)\frac{RI^m_{\alpha,\beta,\gamma,\delta} f(z)}{z} + \lambda (RI^m_{\alpha,\beta,\gamma,\delta} f(z))' - 1}{(1-\lambda)\frac{RI^m_{\alpha,\beta,\gamma,\delta} f(z)}{z} + \lambda (RI^m_{\alpha,\beta,\gamma,\delta} f(z))' + 1 - 2\rho} \right| < \sigma \ , z \in U . \tag{1.8}$$

Definition 1.12: Let $f \in A, m \in N_0 = N \cup \{0\}, \lambda \geq 0, \delta \geq 0, \rho \in [0,1), \sigma \in (0,1], \beta \geq 0, \gamma \geq 0, \ \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\mathrm{H}^m_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma, \rho)$ if and only if

$$\left| \frac{(RI^m_{\alpha,\beta,\gamma,\delta} f(z))' + \lambda z (RI^m_{\alpha,\beta,\gamma,\delta} f(z))'' - 1}{(RI^m_{\alpha,\beta,\gamma,\delta} f(z))' + \lambda z (RI^m_{\alpha,\beta,\gamma,\delta} f(z))'' + 1 - 2\rho} \right| < \sigma \ , z \in U . \tag{1.9}$$

Let T denote the subclass of A consisting of functions whose non-zero coefficients, from second on, are negative; that is, an analytic function f is in T if and only if it can be expressed as

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0, z \in U.$$

If $f \in T$, then $RI^m_{\alpha,\beta,\gamma,\delta} f(z) = z - \sum_{k=2}^{\infty} \zeta_k(\alpha,\beta,\gamma,\delta,m) a_k z^k$, where

$$\zeta_k(\alpha,\beta,\gamma,\delta,m) = (1-\delta)\Omega_k(m) + \delta\Phi_k(\alpha,\beta,\gamma,m), \tag{1.10}$$

$\Omega_k(m)$ and $\Phi_k(\alpha,\beta,\gamma,m)$ are as defined in (1.3) and (1.4), respectively. We denote by $TS^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$, $TK^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$, $TC^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$, $T\mathrm{P}^m_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$ and $T\mathrm{H}^m_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$, the classes of functions $f(z) \in T$ satisfying (1.5), (1.6) (1.7),(1.8) and (1.9) respectively.

In this paper, sharp results concerning coefficients and distortion theorems for the classes $TS^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$, $TK^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$, $TC^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$, $T\mathrm{P}^m_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$ and $T\mathrm{H}^m_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$ are determined. Throughout this paper, unless otherwise mentioned we shall assume that $\zeta_k(\alpha,\beta,\gamma,\delta,m)$ is as defined in (1.10).

2. COEFFICIENT BOUNDS

In this section we study the characterization properties for functions in the classes $TS^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$, $TK^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$, $TC^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$, $T\mathrm{P}^m_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$ and $T\mathrm{H}^m_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$ are determined, following the papers of V. P. Gupta and P. K. Jain [4], [5] and H. Silverman[11].

Theorem 2.1: A function f is in $TS^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$ if and only if

$$\sum_{k=2}^{\infty} \{k - 1 + \sigma(k + 1 - 2\rho)\} \zeta_k(\alpha,\beta,\gamma,\delta,m) a_k \leq 2\sigma(1-\rho). \tag{2.1}$$

The result is sharp.

Proof: Suppose f satisfies (2.1).Then for $|z|<1$, we have

$$\left|z(RI^m_{\alpha,\beta,\gamma,\delta}f(z))' - RI^m_{\alpha,\beta,\gamma,\delta}f(z)\right| - \sigma\left|z(RI^m_{\alpha,\beta,\gamma,\delta}f(z))' + (1-2\rho)RI^m_{\alpha,\beta,\gamma,\delta}f(z)\right|$$

$$= \left|-\sum_{k=2}^{\infty}(k-1)\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k z^k\right| - \sigma\left|2(1-\rho) - \sum_{k=2}^{\infty}(k+1-2\rho)\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k z^k\right|$$

$$\le \sum_{k=2}^{\infty}(k-1)\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k - 2\sigma(1-\rho) + \sum_{k=2}^{\infty}\sigma(k+1-2\rho)\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k$$

$$= \sum_{k=2}^{\infty}\{(k-1+\sigma(k+1-2\rho)\}\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k - 2\sigma(1-\rho) < 0\,.$$

Hence, by using the maximum modulus theorem and (1.5), $f \in TS^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$.

For the converse, assume that

$$\left|\frac{\frac{z(RI^m_{\alpha,\beta,\gamma,\delta}f(z))'}{RI^m_{\alpha,\beta,\gamma,\delta}f(z)} - 1}{\frac{z(RI^m_{\alpha,\beta,\gamma,\delta}f(z))'}{RI^m_{\alpha,\beta,\gamma,\delta}f(z)} + 1 - 2\rho}\right| = \left|\frac{-\sum_{k=2}^{\infty}(k-1)\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k z^k}{2\sigma(1-\rho) - \sum_{k=2}^{\infty}\sigma(k+1-2\rho)\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k z^k}\right| < \sigma\,, z \in U.$$

Since $\operatorname{Re}(z) \le |z|$ for all $z \in U$, we obtain

$$\operatorname{Re}\left(\frac{\sum_{k=2}^{\infty}(k-1)\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k z^k}{2\sigma(1-\rho) - \sum_{k=2}^{\infty}\sigma(k+1-2\rho)\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k z^k}\right) < \sigma\,. \tag{2.2}$$

Choose values of z on the real axis so that $\left(z(RI^m_{\alpha,\beta,\gamma,\delta}f(z)' / RI^m_{\alpha,\beta,\gamma,\delta}f(z)\right)$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1$ through real values, we have the desired inequality (2.1).The function

$$f_1(z) = z - \frac{2\sigma(1-\rho)}{(k-1+\sigma(k+1-2\rho))\zeta_k(\alpha,\beta,\gamma,\delta,m)} z^k, k \ge 2\,, \tag{2.3}$$

is an extremal function for the theorem.

Theorem 2.2: i) A function f is in $TK^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$ if and only if

$$\sum_{k=2}^{\infty}(k+1)(k-1+\sigma(k+1-2\rho))\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k \le 4\sigma(1-\rho)\,. \tag{2.4}$$

ii) A function f is in $TC^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$ if and only if

$$\sum_{k=2}^{\infty}k(k-1+\sigma(k+1-2\rho))\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k \le 2\sigma(1-\rho)\,. \tag{2.5}$$

The results (2.4) and (2.5) are sharp.

The proof of Theorem 2.2 is similar to that of Theorem 2.1 and so omitted. Extremal functions are given by

$$f_2(z) = z - \frac{4\sigma(1-\rho)}{(k+1)(k-1+\sigma(k+1-2\rho))\zeta_k(\alpha,\beta,\gamma,\delta,m)} z^k, k \ge 2\,, \tag{2.6}$$

and

$$f_3(z) = z - \frac{2\sigma(1-\rho)}{k(k-1+\sigma(k+1-2\rho))\zeta_k(\alpha,\beta,\gamma,\delta,m)} z^k, k \ge 2, \tag{2.7}$$

respectively.

Theorem 2.3: i) A function $f(z) \in TP^m_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$ if and only if

$$\sum_{k=2}^{\infty}(1+\lambda(k-1))(1+\sigma)\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k \le 2\sigma(1-\rho)\,. \tag{2.8}$$

ii) A function $f(z) \in TH^{m}_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$ if and only if

$$\sum_{k=2}^{\infty} k(1+\lambda(k-1))(1+\sigma)\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k \le 2\sigma(1-\rho). \tag{2.9}$$

The results (2.8) and (2.9) are sharp.

The proof of Theorem 2.3 is similar to that of Theorem 2.1 and so omitted. Extremal functions are given by

$$f_4(z) = z - \frac{2\sigma(1-\rho)}{(1+\lambda(k-1))(1+\sigma)\zeta_k(\alpha,\beta,\gamma,\delta,m)} z^k, k \ge 2, \tag{2.10}$$

and

$$f_5(z) = z - \frac{2\sigma(1-\rho)}{k(1+\lambda(k-1))(1+\sigma)\zeta_k(\alpha,\beta,\gamma,\delta,m)} z^k, k \ge 2, \tag{2.11}$$

respectively.

Corollary 2.4: i) If $f \in TS^{m}_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$ then $a_k \le \dfrac{2\sigma(1-\rho)}{(k-1+\sigma(k+1-2\rho))\zeta_k(\alpha,\beta,\gamma,\delta,m)}, k \ge 2,$ with equality only for the functions of the form $f_1(z)$, which is as defined in (2.3).

ii) If $f \in TK^{m}_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$,then $a_k \le \dfrac{4\sigma(1-\rho)}{(k+1)(k-1+\sigma(k+1-2\rho))\zeta_k(\alpha,\beta,\gamma,\delta,m)}, k \ge 2,$ with equality only for the functions of the form $f_2(z)$, which is as defined in (2.6).

iii) If $f \in TC^{m}_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$,then $a_k \le \dfrac{2\sigma(1-\rho)}{k(k-1+\sigma(k+1-2\rho))\zeta_k(\alpha,\beta,\gamma,\delta,m)}, k \ge 2,$ with equality only for the functions of the form $f_3(z)$, which is as defined in (2.7).

iv) If $f(z) \in TP^{m}_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$,then $a_k \le \dfrac{2\sigma(1-\rho)}{(1+\lambda(k-1))(1+\sigma)\zeta_k(\alpha,\beta,\gamma,\delta,m)}, k \ge 2,$ with equality only for the functions of the form $f_4(z)$, which is as defined in (2.10).

v) If $f(z) \in TH^{m}_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$, then $a_k \le \dfrac{2\sigma(1-\rho)}{k(1+\lambda(k-1))(1+\sigma)\zeta_k(\alpha,\beta,\gamma,\delta,m)}, k \ge 2,$ with equality only for the functions of the form $f_5(z)$, which is as defined in (2.11).

3. DISTORTION THEOREMS

Theorem 3.1: If a function $f(z) \in T$ is in $TS^{m}_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$ then

$$|f(z)| \ge |z| - \frac{2\sigma(1-\rho)}{(1+\sigma(3-2\rho))\zeta_2(\alpha,\beta,\gamma,\delta,m)}|z|^2, z \in U$$

and

$$|f(z)| \le |z| + \frac{2\sigma(1-\rho)}{(1+\sigma(3-2\rho))\zeta_2(\alpha,\beta,\gamma,\delta,m)}|z|^2, z \in U,$$

with equalities for $f(z) = z - \dfrac{2\sigma(1-\rho)}{(1+\sigma(3-2\rho))\zeta_2(\alpha,\beta,\gamma,\delta,m)} z^2$,(z$\pm r$).

Proof: In view of Theorem 2.1, we have

$$(1+\sigma(3-2\rho))\zeta_2(\alpha,\beta,\gamma,\delta,m)\sum_{k=2}^{\infty} a_k \le \sum_{k=2}^{\infty}(k-1+\sigma(k+1-2\rho)\zeta_k(\alpha,\beta,\gamma,\delta,m)a_k \le 2\sigma(1-\rho).$$

Thus $\displaystyle\sum_{k=2}^{\infty} a_k \le \frac{2\sigma(1-\rho)}{(1+\sigma(3-2\rho))\zeta_2(\alpha,\beta,\gamma,\delta,m)}$ .So we get for $z \in U$,

$$\left|f(z)\right| \le |z| + |z|^2 \sum_{k=2}^{\infty} a_k \le |z| + \frac{\sigma(1-\rho)}{(1+\sigma(3-2\rho))\zeta_2(\alpha,\beta,\gamma,\delta,m)\}}|z|^2 .$$

On the other hand

$$\left|f(z)\right| \ge |z| - |z|^2 \sum_{k=2}^{\infty} a_k \ge |z| - \frac{2\sigma(1-\rho)}{(1+\sigma(3-2\rho))\zeta_2(\alpha,\beta,\gamma,\delta,m)}|z|^2 .$$

Theorem 3.2: i) If a function $f(z) \in T$ is in $T\ell^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$ then

$$\left|f(z)\right| \ge |z| - \frac{4\sigma(1-\rho)}{3(1+\sigma(3-2\rho)\zeta_2(\alpha,\beta,\gamma,\delta,m)}|z|^2, z \in U$$

and

$$\left|f(z)\right| \le |z| + \frac{4\sigma(1-\rho)}{3(1+\sigma(3-2\rho)\zeta_2(\alpha,\beta,\gamma,\delta,m)\}}|z|^2, z \in U .$$

ii) If a function $f(z) \in T$ is in $T\Re^m_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$ then

$$\left|f(z)\right| \ge |z| - \frac{\sigma(1-\rho)}{(1+\sigma(3-2\rho))\zeta_2(\alpha,\beta,\gamma,\delta,m)}|z|^2, z \in U$$

and

$$\left|f(z)\right| \le |z| + \frac{\sigma(1-\rho)}{(1+\sigma(3-2\rho))\zeta_2(\alpha,\beta,\gamma,\delta,m)\}}|z|^2, z \in U .$$

The proof of Theorem 3.2 is similar to that of Theorem 3.1.

Remark 3.3: The bounds of Theorem 3.2 are sharp since the equalities are attained for the functions $f(z) = z - \frac{4\sigma(1-\rho)}{3(1+\sigma(3-2\rho))\zeta_2(\alpha,\beta,\gamma,\delta,m)} z^2 \quad (z = \pm r)$ and

$$f(z) = z - \frac{\sigma(1-\rho)}{(1+\sigma(3-2\rho))\zeta_2(\alpha,\beta,\gamma,\delta,m)} z^2 \ (z = \pm r), \text{ respectively}$$

Theorem 3.4: i) If a function $f(z) \in T$ is in $T\mathrm{P}^m_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$ then

$$\left|f(z)\right| \ge |z| - \frac{2\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\zeta_2(\alpha,\beta,\gamma,\delta,m)\}}|z|^2, z \in U$$

and

$$\left|f(z)\right| \le |z| + \frac{2\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\zeta_2(\alpha,\beta,\gamma,\delta,m)}|z|^2, z \in U .$$

ii) If a function $f(z) \in T$ is in $T\mathrm{H}^m_{\alpha,\beta,\gamma,\lambda,\delta}(\sigma,\rho)$ then

$$\left|f(z)\right| \ge |z| - \frac{\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\zeta_2(\alpha,\beta,\gamma,\delta,m)}|z|^2, z \in U$$

and

$$\left|f(z)\right| \le |z| + \frac{\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\zeta_2(\alpha,\beta,\gamma,\delta,m)}|z|^2, z \in U .$$

The proof of Theorem 3.2 is similar to that of Theorem 3.1.

Remark 3.5: The bounds of Theorem 3.2 are sharp since the equalities are attained for the functions

$$f(z) = z - \frac{2\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\zeta_2(\alpha,\beta,\gamma,\delta,m)\}} z^2 \quad (z = \pm r) \text{ and}$$

$$f(z) = z - \frac{\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\zeta_2(\alpha,\beta,\gamma,\delta,m)\}} z^2 \ (z = \pm r), \text{ respectively}$$

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