

NONSINGULAR PQ-INJECTIVE MODULES

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ABSTRACT

Let M be a right R -module. A right R -module N is called nonsingular principally M -injective (briefly, nonsingular PM -injective) if, for each $m \in M \setminus Z(M)$, any R -homomorphism from mR to N can be extended to an R -homomorphism from M to N . M is called nonsingular principally quasi-injective (briefly, nonsingular PQ -injective) if, it is nonsingular PM -injective. In this paper, we give some characterizations and properties of nonsingular PQ -injective modules.

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1. INTRODUCTION

Let R be a ring. A right R -module M is called *principally injective* (or *P -injective*), if every R -homomorphism from a principal right ideal of R to M can be extended to an R -homomorphism from R to M . Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$ where l and r are left and right annihilators, respectively. This notion was introduced by Camillo [2] for commutative rings. In [8], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. Nicholson, Park, and Yousif [9] extended this notion of principally injective rings to the one for modules. In [5], W. Junchao introduced the definition of Jcp -injective rings, a ring R is called right Jcp -injective if for each $a \in R \setminus Z_r$, any R -homomorphism from aR to R can be extended to an R -homomorphism from R to R . A right R -module M is called *almost mininjective* [11] if, for any simple right ideal kR of R , there exists an S -submodule X_k of M such that $l_M(r_R(m)) = Mk \oplus X_k$ as left S -modules. A ring R is called *right almost mininjective* if R_R is almost mininjective. In this note we introduce the definition of nonsingular PQ -injective modules and give some characterizations and properties.

Throughout this paper, R will be an associative ring with identity and all modules are unitary right R -modules. For right R -modules M and N , $\text{Hom}_R(M, N)$ denotes the set of all R -homomorphisms from M to N and $S = \text{End}_R(M)$ denotes the endomorphism ring of M . If X is a subset of M the right (resp. left) annihilator of X in R (resp. S) is denoted by $r_R(X)$ (resp. $l_S(X)$). By notation $N \subset^\oplus M$ ($N \subset^e M$) we mean that N is a direct summand (an essential submodule) of M . We denote the singular submodule of M by $Z(M)$.

2. NONSINGULAR PM -INJECTIVE MODULES

Recall that a submodule K of a right R -module M is *essential* (or *large*) in M if, every nonzero submodule L of M , we have $K \cap L \neq 0$. An element $m \in M$ is called *singular* if $r_R(m) \subset^e R$. M is called *nonsingular* if it contains no nontrivial singular element.

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Definition 2.1: Let M be a right R -module. A right R -module N is called *nonsingular principally M -injective* (briefly, *nonsingular PM-injective*) if, for each $m \in M \setminus Z(M)$, any R -homomorphism from mR to N can be extended to an R -homomorphism from M to N .

Lemma 2.2: Let M and N be right R -modules. Then N is nonsingular PM -injective if and only if for each $m \in M \setminus Z(M)$,

$$\text{Hom}_R(M, N)m = I_N r_R(m).$$

Proof: Clearly, $\text{Hom}_R(M, N)m \subset I_N r_R(m)$.

Let $x \in I_N r_R(m)$. Define $\varphi : mR \rightarrow xR$ by $\varphi(mr) = xr$ for every $r \in R$. Since $r_R(m) \subset r_R(x)$, φ is well-defined. It is clear that φ is an R -homomorphism. Since N is nonsingular PM -injective, there exists an R -homomorphism $\hat{\varphi} : M \rightarrow N$ such that $\hat{\varphi}\iota_1 = \iota_2\varphi$, where $\iota_1 : mR \rightarrow M$ and $\iota_2 : xR \rightarrow N$ are the inclusion maps. Hence $x = \varphi(m) = \hat{\varphi}(m) \in \text{Hom}_R(M, N)m$.

Conversely, let $m \in M \setminus Z(M)$, and $\varphi : mR \rightarrow N$ be an R -homomorphism. Then $\varphi(m) \in I_N r_R(m)$ so by assumption, we have $\varphi(m) = \hat{\varphi}(m)$ for some $\hat{\varphi} \in \text{Hom}_R(M, N)$. This shows that N is nonsingular PM -injective.

Example 2.3: Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field.

(1) If $M_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ and $N_R = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, then N is not nonsingular PM -injective.

(2) If $M_R = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, then N is nonsingular PM -injective.

Proof: (1) It is clear that only $\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ are nonzero nonessential principal right ideals of

R . Let $m = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in M$ with $x \neq 0$ or $y \neq 0$. Then $m \in M \setminus Z(M)$ and that nonzero submodules mR of

M may be $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ or M . It is clear that $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. For any R -homomorphism

$$\varphi : \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \text{ with } \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \text{ for some } x \in F,$$

$$\varphi\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for every } \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, \text{ hence } \varphi = 0.$$

Then N is not nonsingular PM -injective.

(2) For $M_R = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, let $m = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \in M$ where $x \neq 0$ or $y \neq 0$. Then

$r_R(m) = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ is a nonessential right ideal of R and mR may be $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$ or M .

Let $\alpha : \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ be an R -homomorphism.

Then there exists $x_1, x_2 \in F$ such that $\alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$.

Hence

$$\begin{aligned} \alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) &= \alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_2 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

It follows that $x_1 = 0$.

Define $\hat{\alpha}: M \rightarrow N$ by $\hat{\alpha}\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & ax_2 \\ 0 & 0 \end{pmatrix}$ for every $a, b \in F$.

It is clear that $\hat{\alpha}$ is an R -homomorphism and

$$\hat{\alpha}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \hat{\alpha}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_2 \\ 0 & 0 \end{pmatrix}. \text{ This shows that } \hat{\alpha} \text{ is an extension of } \alpha.$$

Then N is nonsingular PM -injective.

Lemma 2.4:

- (1) N is nonsingular PM -injective if and only if N is nonsingular PX -injective for any submodule X of M .
- (2) $\bigoplus_{i=1}^n N_i$ is nonsingular PM -injective if and only if N_i is nonsingular PM - injective for all i .
- (3) If $m \in M \setminus Z(M)$ and mR is nonsingular PM -injective, then $mR \subset^{\oplus} M$.

Proof:

- (1) The sufficiency is trivial. For the necessity, let $x \in X \setminus Z(X)$, and $\varphi: xR \rightarrow N$ be an R -homomorphism. Since $x \in M \setminus Z(M)$, there exists an R -homomorphism $\hat{\varphi}: M \rightarrow N$ such that $\varphi = \hat{\varphi}\iota_2\iota_1$ where $\iota_1: xR \rightarrow X$ and $\iota_2: X \rightarrow M$ are the inclusion maps. Then $\hat{\varphi}\iota_2$ extends φ .
- (2) The necessity is trivial. For the sufficiency, let $m \in M \setminus Z(M)$, and $\varphi: mR \rightarrow \bigoplus_{i=1}^n N_i$ be an R -homomorphism. Then for each i , there exists R -homomorphisms $\varphi_i: M \rightarrow N_i$ such that $\varphi_i\iota = \pi_i\varphi$ where $\pi_i: \bigoplus_{i=1}^n N_i \rightarrow N_i$ is the projection map, and $\iota: mR \rightarrow M$ is the inclusion map. Put $\hat{\varphi} = \iota_1\varphi_1 + \dots + \iota_n\varphi_n: M \rightarrow \bigoplus_{i=1}^n N_i$. Then it is clear that $\hat{\varphi}$ extends φ .
- (3) Since mR is nonsingular PM -injective, there exists an R -homomorphism $\varphi: M \rightarrow mR$ such that $\varphi\iota = 1_{mR}$ where $\iota: mR \rightarrow M$ is the inclusion map. Then by [1, Lemma 5.1], ι is a split monomorphism, therefore $mR \subset^{\oplus} M$.

Theorem 2.5: The following conditions are equivalent for a projective module M .

- (1) Every $m \in M \setminus Z(M)$, mR is projective.
- (2) Every factor module of a nonsingular PM -injective module is nonsingular PM - injective.
- (3) Every factor module of an injective R -module is nonsingular PM -injective.

Proof:

(1) \Rightarrow (2): Let N be a nonsingular PM -injective module, X a submodule of N , $m \in M \setminus Z(M)$, and $\varphi: mR \rightarrow N/X$ be an R -homomorphism. Then by (1), there exists an R -homomorphism $\hat{\varphi}: mR \rightarrow N$ such that $\varphi = \eta\hat{\varphi}$ where $\eta: N \rightarrow N/X$ is the natural R -epimorphism. Since N is nonsingular PM -injective, there exists an R -homomorphism $\beta: M \rightarrow N$ which is an extension of $\hat{\varphi}$ to M . Then $\eta\beta$ is an extension of φ to M .

(2) \Rightarrow (3): is clear.

(3) \Rightarrow (1): Let $m \in M \setminus Z(M)$, $h: A \rightarrow B$ an R -epimorphism, and let $\alpha: mR \rightarrow B$ be an R -homomorphism. Embed A in an injective module E [1, 18.6]. Let $\sigma: B \rightarrow A/\text{Ker}(h)$ be an R -isomorphism. Since $E/\text{Ker}(h)$ is nonsingular PM-injective, there exists an R -homomorphism $\hat{\alpha}: M \rightarrow E/\text{Ker}(h)$ such that $\iota_1 \sigma \alpha = \hat{\alpha} \iota_2$ where $\iota_1: A/\text{Ker}(h) \rightarrow E/\text{Ker}(h)$ and $\iota_2: mR \rightarrow M$ are the inclusion maps. Since M is projective, $\hat{\alpha}$ can be lifted to $\beta: M \rightarrow E$. Let $x \in mR$. Then $\sigma \alpha(x) = a + \text{Ker}(h)$ for some $a \in A$, so $\beta(x) + \text{Ker}(h) = \eta \beta(x) = \hat{\alpha}(x) = \sigma \alpha(x) = a + \text{Ker}(h)$ where $\eta: E \rightarrow E/\text{Ker}(h)$ is the natural R -epimorphism. Hence $\beta(x) - a \in \text{Ker}(h) \subset A$ so $\beta(x) \in A$. This shows that $\beta(mR) \subset A$. Therefore we have lifted α .

3. NONSINGULAR PQ-INJECTIVE MODULES

A right R -module M is called *nonsingular principally quasi-injective* (briefly, *nonsingular PQ-injective*) if, it is nonsingular PM-injective.

Lemma 3.1: Let M be a right R -module and $S = \text{End}_R(M)$. Then the following conditions are equivalent.

- (1) M is nonsingular PQ-injective.
- (2) $l_M r_R(m) = Sm$ for each $m \in M \setminus Z(M)$.
- (3) $r_R(m) \subset r_R(n)$, where $m, n \in M$ with $m \in M \setminus Z(M)$, implies that $Sn \subset Sm$.
- (4) $l_M(r_R(m) \cap aR) = l_M(a) + Sm$ for all $a \in R$ and $m \in M$ with $ma \in M \setminus Z(M)$.
- (5) If $\alpha: mR \rightarrow M$ is an R -homomorphism, $m \in M \setminus Z(M)$, then $\alpha(m) \in Sm$.

Proof:

(1) \Leftrightarrow (2): by Lemma 2.2

(2) \Rightarrow (3): If $r_R(m) \subset r_R(n)$, where $m, n \in M$ with $m \in M \setminus Z(M)$, then $l_M r_R(n) \subset l_M r_R(m)$. Then $Sn \subset l_M r_R(n) \subset l_M r_R(m) = Sm$ by (2).

(3) \Rightarrow (4): Let $a \in R$ and $m \in M$ with $ma \in M \setminus Z(M)$ and let $x \in l_M(r_R(m) \cap aR)$. Then $r_R(ma) \subset r_R(xa)$, and hence by (3), $Sxa \subset Sma$. Thus $xa = \varphi(ma)$, $\varphi \in S$ and so $(x - \varphi(m)) \in l_M(a)$. It follows that $x \in l_M(a) + Sm$. The other hand is clear.

(4) \Rightarrow (5): Put $a = 1_R$ in (4), then $\alpha(m) \in l_M r_R(m) = l_M(r_R(m) \cap 1R) = l_M(1_R) + Sm = Sm$ because $m1 \in M \setminus Z(M)$.

(5) \Rightarrow (1): Let $m \in M$ with $m \in M \setminus Z(M)$ and let $\varphi: mR \rightarrow M$ be an R -homomorphism. Then by (5), $\varphi(m) \in Sm$ so there exists an R -homomorphism $\hat{\varphi} \in S$ is an extension of φ to M .

Theorem 3.2: Let M be a nonsingular PQ-injective module and $m, n \in M$ with $m \in M \setminus Z(M)$.

- (1) If mR embeds into nR , then Sm is an image of Sn .
- (2) If nR is an image of mR , then Sn can be embedded into Sm .
- (3) If $mR \simeq nR$, then $Sm \simeq Sn$.

Proof:

- (1) Let $\sigma: mR \rightarrow nR$ be an R -monomorphism and let $\iota_1: mR \rightarrow M$ and $\iota_2: nR \rightarrow M$ be the inclusion maps. Since M is nonsingular PQ-injective, there exists an R -homomorphism $\hat{\sigma}: M \rightarrow M$ such that $\hat{\sigma} \iota_1 = \iota_2 \sigma$. Let $\varphi: Sn \rightarrow Sm$ defined by $\varphi(\alpha(n)) = \alpha \hat{\sigma}(m)$ for every $\alpha \in S$. Since $\varphi(\alpha(n)) = \alpha(\hat{\sigma}(m)) = \alpha(\sigma(m)) \in \alpha(nR)$, φ is well-defined. It is clear that φ is an S -homomorphism.

Since σ is monic, $r_R(\sigma(m)) \subset r_R(m)$ and $\sigma(m) \in M \setminus Z(M)$ and hence by Lemma 3.1, $S\sigma(m) \subset S\sigma(m)$. Then $m \in S\sigma(m) \subset \varphi(Sn)$.

- (2) By the same notations as in (1), let $\sigma : mR \rightarrow nR$ be an R -epimorphism. Write $\sigma(ms) = n$, $s \in R$. Since M is nonsingular PQ-injective, σ can be extended to $\hat{\sigma} : M \rightarrow M$ such that $\hat{\sigma}_1 = \iota_2 \sigma$. Define $\varphi : Sn \rightarrow Sm$ by $\varphi(\alpha(n)) = \alpha \hat{\sigma}(ms)$ for every $\alpha \in S$. It is clear that φ is an S -homomorphism. If $\alpha(n) \in \text{Ker}(\varphi)$, then $0 = \varphi(\alpha(n)) = \alpha \hat{\sigma}(ms) = \alpha(n)$. This shows that φ is an S -monomorphism.
- (3) Follows from (1) and (2).

Recall that a right R -module M is called C2 [6] if, every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M . M is called C3 if, whenever N and K are direct summands of M with $N \cap K = 0$ then $N \oplus K$ also a direct summand of M .

Theorem 3.3: Let $M = mR$ be a principle, nonsingular PQ-injective module.

- (1) If $X = e(mR)$, $e^2 = e \in S$ and $e(m) \in M \setminus Z(M)$, then $X = g(mR)$, for some $g^2 = g \in S$.
- (2) If $e(mR) \cap f(mR) = 0$, $e^2 = e \in S$, $f^2 = f \in S$ and $f(m) \in M \setminus Z(M)$, then $e(M) \oplus f(M) = g(M)$, for some $g^2 = g \in S$.

Proof:

- (1) Let $\sigma : e(mR) \rightarrow X$ be an R -isomorphism. Write $\sigma e(m) = x$ where $x \in X$ so $xR = X$. We must show that $xR = g(mR)$, for some $g^2 = g \in S$. Then by Lemma 2.3, we have $e(mR)$ is nonsingular PM-injective and hence xR is also nonsingular PM-injective. Since $\sigma e(m) \in M \setminus Z(M)$, $xR \subset^{\oplus} mR$ Lemma 2.4.
- (2) Let $e(mR) \cap f(mR) = 0$, $e^2 = e \in S$, $f^2 = f \in S$ and $f(m) \in M \setminus Z(M)$. Then $e(mR) \oplus f(mR) = e(mR) \oplus (1-e)f(mR)$. If $(1-e)f(mR) = 0$, then $e(mR) \oplus f(mR)$ is a direct summand of M . If $(1-e)f(mR) \neq 0$, then $(1-e)f(mR) = f(mR)$, and hence $(1-e)f(mR) = g(mR)$ for some $g^2 = g \in S$ by (1). Let $h = e + g - ge$, then $h^2 = h$ and $e(M) \oplus f(M) = h(M)$.

Lemma 3.4: Let M be a nonsingular PQ-injective module and $S = \text{End}_R(M)$. If $\alpha \in S$ with $\alpha(M) \subset M \setminus Z(M)$, then $I_S(\text{Ker}(\alpha) \cap mR) = I_S(m) + S\alpha$.

Proof: Clearly, $I_S(m) + S\alpha \subset I_S(\text{Ker}(\alpha) \cap mR)$. Let $\beta \in I_S(\text{Ker}(\alpha) \cap mR)$. Then $r_R(\alpha(m)) \subset r_R(\beta(m))$, so $I_M r_R(\beta(m)) \subset I_M r_R(\alpha(m))$. Since $\alpha(m) \in M \setminus Z(M)$, $S\beta(m) \subset I_M r_R(\beta(m)) \subset I_M r_R(\alpha(m)) = S\alpha(m)$ by Lemma 3.1, so $\beta(m) = s\alpha(m)$ for some $s \in S$. It follows that $(\beta - s\alpha) \in I_S(m)$, and hence $\beta \in I_S(m) + S\alpha$.

Following [8], a right R -module M is called a *principal self-generator*, if every element $m \in M$ has the form $m = \gamma(m_1)$ for some $\gamma : M \rightarrow mR$. If $uR \neq 0$ is uniform, we call u a *uniform element* of M . We call a right R -module M is a *duo module* if every submodule of M is fully invariant.

Theorem 3.5: Let M be a principal module which is a principal self-generator. Then the following conditions are equivalent.

- (1) M is nonsingular PQ-injective.
- (2) $I_S(\text{Ker}(\alpha) \cap mR) = I_S(m) + S\alpha$ for all $m \in M$ and $\alpha \in S$ with $\alpha(M) \in M \setminus Z(M)$.
- (3) $I_S(\text{Ker}(\alpha)) = S\alpha$ for all $\alpha \in S$ with $\alpha(M) \in M \setminus Z(M)$.
- (4) $\text{Ker}(\alpha) \subset \text{Ker}(\beta)$, where $\alpha, \beta \in S$ with $\alpha(m) \in M \setminus Z(M)$, implies that $S\beta \subset S\alpha$.

Proof:

- (1) \Rightarrow (2) : by Lemma 3.4.
- (2) \Rightarrow (3) : If $M = m_0R$, take $m = m_0$ in (2).

(3) \Rightarrow (4) : $\text{Ker}(\alpha) \subset \text{Ker}(\beta)$, then $I_S(\text{Ker}(\beta)) \subset I_S(\text{Ker}(\alpha))$. It follows that

$$S\beta \subset I_S(\text{Ker}(\beta)) \subset I_S(\text{Ker}(\alpha)) = S\alpha.$$

(4) \Rightarrow (1) : Let $m \in M \setminus Z(M)$, $\varphi : mR \rightarrow M$ be an R -homomorphism.

Since M is a principal self-generator, there exists $\beta \in S$ such that $\beta(m_1) = m$, so $\text{Ker}(\beta) \subset \text{Ker}(\varphi\beta)$ and $\beta(M) \subset M \setminus Z(M)$. Then by (4), $S\varphi\beta \subset S\beta$ hence $\varphi\beta = \hat{\varphi}\beta$ for some $\hat{\varphi} \in S$. This shows that $\hat{\varphi}$ is an extension of φ .

Theorem 3.6: Let M be a duo, nonsingular PQ-injective module. If u a uniform element of M with $u \in M \setminus Z(M)$, then $M_u = \{\alpha \in S \mid \text{Ker}(\alpha) \cap uR \neq 0\}$ is a unique maximal left ideal of S containing $I_S(u)$.

Proof: Since uR is uniform, M_u is a left ideal of S . It is clear that $I_S(u) \subset M_u \neq S$. Let X be a left ideal of S containing $I_S(u)$ and $X \neq S$. If $\alpha \in X - M_u$, then $\text{Ker}(\alpha) \cap uR = 0$. Since M is a duo module, $\alpha(u)R \subset M \setminus Z(M)$ and by Lemma 3.4 we have $S = I_S(\text{Ker}(\alpha) \cap uR) = I_S(u) + S\alpha \subset X$ a contradiction. Thus $X \subset M_u$.

Definition 3.7: Let M be a right R -module, $S = \text{End}_R(M)$. The module M is called *almost nonsingular PQ-injective* if, for each $m \in M \setminus Z(M)$, there exists an S -submodule X_m of M such that $I_M(r_R(m)) = Sm \oplus X_m$ as left S -modules.

Theorem 3.8: Let M be a right R -module, $S = \text{End}_R(M)$ and $m \in M \setminus Z(M)$.

- (1) If $\text{Hom}_R(mR, M) = S \oplus Y$ as left S -modules, then $I_M(r_R(m)) = Sm \oplus X$ as left S -modules, where $X = \{f(m) : f \in Y\}$.
- (2) If $I_M(r_R(m)) = Sm \oplus X$ for some $X \subset M$ as left S -modules, then we have $\text{Hom}_R(mR, M) = S \oplus Y$ as left S -modules, where $Y = \{f \in \text{Hom}_R(mR, M) : f(m) \in X\}$.
- (3) Sm is a direct summand of $I_M(r_R(m))$ as left S -modules if and only if S is a direct summand of $\text{Hom}_R(mR, M)$ as left S -modules.

Proof: Define $\theta : \text{Hom}_R(mR, M) \rightarrow I_M(r_R(m))$ by $\theta(f) = f(m)$ for every $f \in \text{Hom}_R(mR, M)$. It is obvious that θ is an S -monomorphism. For $x \in I_M(r_R(m))$, define $g : mR \rightarrow M$ by $g(mr) = xr$ for every $r \in R$. Since $r_R(m) \subset r_R(x)$, g is well-defined, so it is clear that g is an R -homomorphism. Then $\theta(g) = g(m) = x$. Therefore θ is an S -isomorphism. Let $\alpha(m) \in Sm$. Since $\alpha(m) \in I_M(r_R(m))$, there exists $\varphi \in \text{Hom}_R(mR, M)$ such that $\theta(\varphi) = \alpha(m)$, so $\varphi(m) = \alpha(m)$. Define $\hat{\varphi} : M \rightarrow M$ by $\hat{\varphi}(x) = \alpha(x)$ for every $x \in M$. It is clear that $\hat{\varphi}$ is an R -homomorphism and is an extension of φ . Then $\alpha(m) = \hat{\varphi}(m) = \theta(\hat{\varphi})$. This shows that $Sm \subset \theta(S)$. The other inclusion is clear. Then $\theta(S) = Sm$ and $X = \theta(Y) = \{f(m) : f \in Y\}$. Then the Lemma follows.

Theorem 3.9: The following conditions are equivalent:

- (1) M is almost nonsingular PQ-injective.
- (2) There exists an indexed set $\{X_m : m \in M\}$ of S -submodules of M with the property that if $mR \subset M \setminus Z(M)$, $m \in M$, then $I_M(r_R(m) \cap aR) = (X_{ma} : a)_1 + Sm$ and $(X_{ma} : a)_1 \cap Sm \subset I_M(a)$ for all $a \in R$, where $(X_{ma} : a)_1 = \{n \in M : na \in X_{ma}\}$ if $ma \neq 0$ and $(X_{ma} : a)_1 = I_M(aR)$ if $ma = 0$.

Proof:

(1) \Rightarrow (2) Let $m \in M \setminus Z(M)$. Then there exists an S -submodule X_m of M such that $I_M(r_R(m)) = Sm \oplus X_m$ as left S -modules. Let $a \in R$. If $ma = 0$, then $aR \subset r_R(m)$ so (2) follows. If $ma \neq 0$, then any $x \in I_M(r_R(m)) \cap aR$ we have $r_R(ma) \subset r_R(xa)$ and so $xa \in I_M(r_R(xa)) \subset I_M(r_R(ma)) = Sma \oplus X_{ma}$ because $ma \in M \setminus Z(M)$. Write $xa = \alpha(ma) + y$ where $\alpha \in S$ and $y \in X_{ma}$. Then $(x - \alpha(m))a = y \in X_{ma}$, so $x - \alpha(m) \in (X_{ma} : a)_1$. It follows that $x \in (X_{ma} : a)_1 + Sm$.

This shows that $I_M(r_R(m) \cap aR) \subset (X_{ma} : a)_1 + Sm$. Conversely, it is clear that $Sm \subset I_M(r_R(m) \cap aR)$. Let $y \in (X_{ma} : a)_1$. Then $ya \in X_{ma} \subset I_M(r_R(ma))$. If $as \in r_R(m) \cap aR$, then $mas = 0$ and so $yas = 0$. Hence $y \in I_M(r_R(m) \cap aR)$. This shows that $(X_{ma} : a)_1 \subset I_M(r_R(m) \cap aR)$. Therefore $I_M(r_R(m) \cap aR) = (X_{ma} : a)_1 + Sm$. If $\beta(m) \in (X_{ma} : a)_1 \cap Sm$, then $\beta(m)a \in X_{ma} \cap Sma = 0$. Hence $\beta(m) \in I_M(a)$.

(2) \Rightarrow (1) Let $m \in M \setminus Z(M)$. Then there exists an S -submodule X_m of M such that $I_M(r_R(m)) = I_M(r_R(m) \cap R) = (X_m : 1)_1 + Sm$ and $(X_m : 1)_1 \cap Sm \subset I_M(1) = 0$. Note that $(X_m : 1)_1 = X_m$. Then (1) follows.

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