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# CERTAIN COEFFICIENT INEQUALITIES FOR SAKAGUCHI TYPE FUNCTIONS AND APPLICATIONS TO FRACTIONAL DERIVATIVES

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*ABSTRACT*

*In the present paper, sharp upper bounds of* $\left|a_3 - \mu\, a_2^2\right|$ *for the functions* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *belonging to a new subclass of Sakaguchi type functions are obtained. Also, application of our results for subclass of functions defined by convolution with a normalized analytic function are given. In particular, Fekete-Szegö inequalities for certain classes of functions defined through fractional derivatives are obtained.*

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## 1. Introduction

Let $\mathcal{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta := \{z \in C / |z| < 1\} \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. For two functions f, g ∈ $\mathcal{A}$, we say that the function f(z) is subordinate to g(z) in Δ and write $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function w(z) with w(0) = 0 and |w(z)| < 1 (z ∈ Δ), such that f(z) = g(w(z)), (z ∈ Δ). In particular, if the function g is univalent in Δ, the above subordination is equivalent to f(0) = g(0) and f(Δ) ⊂ g(Δ).

A function f(z) ∈ $\mathcal{A}$ is said to be in the class M(λ, α, t) if it satisfies

$$\mathrm{Re}\left\{\frac{(1-t)[\lambda z^2 f''(z) + zf'(z)]}{(1-\lambda)[f(z)-f(tz)] + \lambda z[f'(z) - tf'(tz)]}\right\} > \alpha,$$

$$|t| \le 1, t \ne 1, 0 \le \lambda \le 1, 0 \le \alpha < 1 \tag{1.2}$$

For the case λ = 0 in (1.2) we get a Sakaguchi type class $S^*(\alpha, t)$. A function f(z) ∈ $\mathcal{A}$ is said to be in the class $S^*(\alpha, t)$ if it satisfies

$$\mathrm{Re}\left\{\frac{(1-t)zf'(z)}{f(z)-f(zt)}\right\} > \alpha, \quad |t| \le 1, t \ne 1 \tag{1.3}$$

which was introduced and studied by Owa et al. [9, 10] .For some α ∈ [0, 1) and for all z ∈ Δ. For λ = 0, α = 0 and t = −1 in M(λ, α, t), we get the class the class $S^*(0, -1)$ studied by Sakaguchi [11]. A function f(z) ∈ $S^*(\alpha, -1)$ is called Sakaguchi function of order α.

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In this paper, we define the following class M(λ, ϕ, t), which is generalization of the class M(λ, α, t).

**Definition: 1.1** Let $\phi(z) = 1 + B_1z + B_2z^2 + \cdots$ be univalent starlike function with respect to '1' which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis, and let $B_1 > 0$. Then function f ∈ $\mathcal{A}$ is in the class M(λ, ϕ, t) if

$$\left\{\frac{(1-t)[\lambda z^2 f''(z) + zf'(z)]}{(1-\lambda)[f(z)-f(tz)] + \lambda z[f'(z) - tf'(tz)]}\right\} \prec \phi(z),$$

$$|t| \le 1, t \ne 1, 0 \le \lambda \le 1 \tag{1.4}$$

For λ = 0 in (1.4) we get the class $S^*(\phi, t)$ which was defined by Goyal and Goswami [3].

**Definition: 1.2** Let $\phi(z) = 1 + B_1z + B_2z^2 + \cdots$ be univalent starlike function with respect to '1' which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis, and let $B_1 > 0$. Then function f ∈ $\mathcal{A}$ is in the class $S^*(\phi, t)$ if

$$\left\{\frac{(1-t)zf'(z)}{f(z)-f(zt)}\right\} \prec \phi(z), \quad |t| \le 1, t \ne 1 \tag{1.5}$$

Again T(ϕ, t) denote the subclass of $\mathcal{A}$ consisting functions f(z) such that $zf'(z) \in S^*(\phi, t)$.

When ϕ(z) = (1+Az)/(1+Bz), (−1 ≤ B < A ≤ 1), we denote the subclasses $S^*(\phi, t)$ and T(ϕ, t) by $S^*[A, B, t]$ and $\mathcal{T}[A, B, t]$ respectively.

Obviously $S^*(\phi, 0) \equiv S^*(\phi)$. When t = −1, then $S^*(\phi, -1) \equiv S_S^*(\phi)$, which is a known class studied by Shanmugam et al. [12]. For t = 0 and ϕ(z) = (1+Az)/(1+Bz), (−1 ≤ B < A ≤ 1), the subclass $S^*(\phi, t)$ reduces to the class $S^*[A, B]$ studied by Janowski [4]. For 0 ≤ α < 1 let $S^*(\alpha, t) := S^*[1-2\alpha, -1; t]$, which is a known class studied by Owa et al. [10] Also,for

t = −1 and $\phi(z) = \dfrac{1+(1-2\alpha)z}{1-z}$, our class reduces to a known class S(α, −1) studied by Cho et al. ([1], see also [10]).

In the present paper, we obtain the Fekete-Szegö inequality for the functions in the subclass M (λ,, ϕ, t). We also give application of our results to certain functions defined through convolution (or Hadamard product) and in particular, we consider the class $M^\delta(\lambda, \phi, t)$ defined by fractional derivatives.

To prove our main results, we need the following lemma:

**Lemma: 1.3** [6] If $p(z) = 1 + c_1z + c_2z^2 + \cdots$ is an analytic function with positive real part in Δ, then

$$|c_2 - vc_1^2| \le \begin{cases} -4v+2 & \text{if } v \le 0, \\ 2 & \text{if } 0 \le v \le 1, \\ 4v-2 & \text{if } v \ge 1. \end{cases}$$

When v < 0 or v > 1, the equality holds if and only if p(z) is (1 + z)/(1 − z) or one of its rotations. If 0 < v < 1, then the equality holds if and only if p(z) is $(1 + z^2)/(1 - z^2)$ or one of its rotations. If v = 0, the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1+z}{1-z} \quad (0 \le \lambda \le 1)$$

or one of its rotations. If v = 1, the equality holds if and only if p(z) is the reciprocal of one of the functions such that the equality holds in the case of v = 0.
Also the above upper bound is sharp, and it can be improved as follows when 0<v<1:

 

$$\left|c_2 - vc_1^2\right| + v\left|c_1\right|^2 \le 2 \quad (0 < v \le 1/2)$$

and

$$\left|c_2 - vc_1^2\right| + (1-v)\left|c_1\right|^2 \le 2 \quad (1/2 < v \le 1).$$

**Lemma: 1.4** [5] If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part, then

$$\left|c_2 - \mu c_1^2\right| \le 2 \max\{1, \left|2\mu - 1\right|\},$$

where $\mu$ is complex and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \qquad p(z) = \frac{1+z}{1-z}.$$

**2. MAIN RESULTS**

Our main result is contained in the following theorem:

**Theorem: 2.1** If f(z) given by (1.1) belongs to $M(\lambda, \phi, t)$, then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{1}{(1+2\lambda)(2+t)(1-t)}\left[B_2 + B_1^2\left(\dfrac{1+t}{1-t}\right) - \dfrac{\mu B_1^2(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)}\right] & \text{if } \mu \le \sigma_1 \\ \dfrac{B_1}{(1+2\lambda)(2+t)(1-t)} & \text{if } \sigma_1 \le \mu \le \sigma_2 \\ -\dfrac{1}{(1+2\lambda)(2+t)(1-t)}\left[B_2 + B_1^2\left(\dfrac{1+t}{1-t}\right) - \dfrac{\mu B_1^2(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)}\right] & \text{if } \mu \ge \sigma_2 \end{cases}$$

where

$$\sigma_1 = \frac{(1+\lambda)^2(1-t)}{B_1(1+2\lambda)(2+t)}\left\{-1 + \frac{B_2}{B_1} + \frac{B_1(1+t)}{(1-t)}\right\}$$

and

$$\sigma_2 = \frac{(1+\lambda)^2(1-t)}{B_1(1+2\lambda)(2+t)}\left\{1 + \frac{B_2}{B_1} + \frac{B_1(1+t)}{(1-t)}\right\}.$$

The result is sharp.

**Proof:** Let $f \in M(\lambda, \phi, t)$. Then there exists a Schwarz function $w(z) \in \mathcal{A}$ such that

$$\frac{(1-t)[\lambda z^2 f''(z) + zf'(z)]}{(1-\lambda)[f(z) - f(tz)] + \lambda z[(f'(z) - tf'(tz)]} = \phi(w(z))$$
$$(z \in \Delta; \left|t\right| \le 1, t \ne 1, 0 \le \lambda \le 1) \tag{2.1}$$

If $p_1(z)$ is analytic and has positive real part in $\Delta$ and $p_1(0) = 1$, then

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \Delta). \tag{2.2}$$

From (2.2), we obtain

$$w(z) = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots. \tag{2.3}$$

Let

 

$$p(z)=\frac{(1-t)[\lambda z^2 f''(z)+zf'(z)]}{(1-\lambda)[f(z)-f(tz)]+\lambda z[(f'(z)-tf'(tz)]}=1+b_1z+b_2z^2+\cdots \quad (z\in\Delta), \tag{2.4}$$

which gives

$$\begin{aligned} b_1 &= (1+\lambda)(1-t)a_2 \qquad \text{and} \\ b_2 &= (1+\lambda)^2(t^2-1)a_2^2+(1+2\lambda)(2-t-t^2)a_3. \end{aligned} \tag{2.5}$$

Since $\phi(z)$ is univalent and $p \prec \phi$, therefore using (2.3), we obtain

$$p(z)=\phi(w(z))=1+\frac{B_1c_1}{2}z+\left\{\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)B_1+\frac{1}{4}c_1^2B_2\right\}z^2+\cdots \quad (z\in\Delta), \tag{2.6}$$

Now from (2.4), (2.5) and (2.6), we have

$$(1+\lambda)(1-t)a_2=\frac{B_1c_1}{2},$$

$$(1+\lambda)^2(t^2-1)a_2^2+(1+2\lambda)(2-t-t^2)a_3=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)B_1+\frac{1}{4}c_1^2B_2,$$

$$|t|\le 1,\ t\ne 1,\ 0\le\lambda\le 1.$$

Therefore we have

$$a_3-\mu a_2^2=\frac{B_1}{2(1+2\lambda)(2+t)(1-t)}\left[c_2-vc_1^2\right], \tag{2.7}$$

where

$$v=\frac{1}{2}\left[1-\frac{B_2}{B_1}-B_1\left(\frac{1+t}{1-t}\right)+\mu B_1\frac{(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)}\right].$$

Our result now follows by an application of Lemma 1.3. To shows that these bounds are sharp, we define the functions $K_{\phi_n}$ (n = 2, 3 …) by

$$\frac{(1-t)[\lambda z^2K''_{\phi_n}(z)+zK'_{\phi_n}(z)]}{(1-\lambda)[K_{\phi_n}(z)-K_{\phi_n}(tz)]+\lambda z[K'_{\phi_n}(z)-tK'_{\phi_n}(tz)]}=\phi(z^{n-1}),$$

$$K_{\phi_n}(0)=0=[K_{\phi_n}]'(0)-1$$

and the function $F_\eta$ and $G_\eta$ $(0\le\eta\le 1)$ by

$$\frac{(1-t)[\lambda z^2F''_\eta(z)+zF'_\eta(z)]}{(1-\lambda)[F_\eta(z)-F_\eta(tz)]+\lambda z[F'_\eta(z)-tF'_\eta(tz)]}=\phi\left(\frac{z(z+\eta)}{(1+\eta z)}\right),$$

$$F_\mu(0)=0=[F_\eta]'(0)-1$$

and

$$\frac{(1-t)[\lambda z^2G''_\eta(z)+zG'_\eta(z)]}{(1-\lambda)[G_\eta(z)-G_\eta(tz)]+\lambda z[G'_\eta(z)-tG'_\eta(tz)]}=\phi\left(\frac{-z(z+\eta)}{(1+\eta z)}\right),$$

$$G_\mu(0)=0=[G_\eta]'(0)-1.$$

Obviously the functions $K_{\phi_n}$, $F_\eta$, $G_\eta \in M(\lambda, \alpha, t)$. Also we write $K_\phi := K_{\phi_2}$. If $\mu<\sigma_1$ or $\mu>\sigma_2$, then equality holds if and only if f is $K_\phi$ or one of its rotations. When $\sigma_1<\mu<\sigma_2$, the equality holds if and only if f is $K_{\phi_2}$ or one of its rotations. $\mu = \sigma_1$ then equality holds if and only if f is $F_\eta$ or one of its rotations. $\mu = \sigma_2$ then equality holds if and only if f is $G_\eta$ or one of its rotations.

If $\sigma_1 \le \mu \le \sigma_2$, in view of Lemma 1.3, Theorem 2.1 can be improved.

**Theorem: 2.2** Let f(z) given by (1.1) belongs to M(λ, α, t) and $\sigma_3$ be given by

$$\sigma_3 = \frac{1}{B_1}\left(\frac{(1+\lambda)^2(1-t)}{(1+2\lambda)(2+t)}\right)\left[\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\right]$$

If $\sigma_1 < \mu \le \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{B_1^2}\left[(B_1 - B_2)\left(\frac{(1+\lambda)^2(1-t)}{(1+2\lambda)(2+t)}\right) - B_1^2\left(\frac{(1+\lambda)^2(1+t)}{(1+2\lambda)(2+t)}\right) + \mu B_1^2\right]\left|a_2\right|^2 \le \frac{B_1}{(1+2\lambda)(2+t)(1-t)}.$$

If $\sigma_3 < \mu \le \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{B_1^2}\left[(B_1 + B_2)\left(\frac{(1+\lambda)^2(1-t)}{(1+2\lambda)(2+t)}\right) + B_1^2\left(\frac{(1+\lambda)^2(1+t)}{(1+2\lambda)(2+t)}\right) - \mu B_1^2\right]\left|a_2\right|^2 \le \frac{B_1}{(1+2\lambda)(2+t)(1-t)}.$$

For λ = 0 in Theorem 2.1 we get Fekete-Szegö inequality for functions to be in the class $S^*(\phi, t)$ which was given by Goyal and Goswami [3].

**Theorem: 2.3** If f(z) is given by (1.1) belongs to M(λ, φ, t) then

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{(1+2\lambda)(2+t)(1-t)} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1(1+t)}{(1-t)} - \frac{(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)}\mu B_1\right|\right\}.$$

The result is sharp.

**Proof:** By applying the Lemma 1.4 in (2.7) we get Theorem 2.3. The result is sharp for the functions defined by

$$\frac{(1-t)[\lambda z^2 f''(z) + zf'(z)]}{(1-\lambda)[f(z) - f(tz)] + \lambda z[f'(z) - tf'(tz)]} = \phi(z^2)$$

and

$$\frac{(1-t)[\lambda z^2 f''(z) + zf'(z)]}{(1-\lambda)[f(z) - f(tz)] + \lambda z[f'(z) - tf'(tz)]} = \phi(z).$$

## 3. APPLICATIONS TO FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVES

For two analytic functions $f(z) = z + \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=0}^{\infty} g_n z^n$, their convolution (or Hadamard product) is defined to be the function $(f * g)(z) = z + \sum_{n=0}^{\infty} a_n g_n z^n$ . For a fixed $g \in \mathcal{A}$, let $M^g(\lambda, \phi, t)$ be the class of functions f $\in \mathcal{A}$ for which $(f * g) \in M(\lambda, \phi, t)$.

**Definition: 3.1** Let f(z) be analytic in a simply connected region of the z-plane containing origin. The fractional derivative of f of order δ is defined by

 

$$ {}_0D_z^{\delta}f(z) := \frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z (z-\zeta)^{-\delta} f(\zeta)d\zeta \quad (0 \le \delta < 1), \tag{3.1} $$

where the multiplicity of $(z - \zeta)^{-\delta}$ is removed by requiring that $\log(z - \zeta)$ is real for $(z - \zeta) > 0$.

Using Definition 3.1, Owa and Srivastava (see [7, 8]; see also [13, 14]) introduced a fractional derivative operator $\Omega^{\delta}$ : $\mathcal{A} \to \mathcal{A}$ defined by

$$ (\Omega^{\delta}f)(z) = \Gamma(2-\delta)z^{\delta}{}_0D_z^{\delta}f(z), \quad (\delta \ne 2,3,4,...). $$

The class $M^{\delta}(\lambda, \phi, t)$ consists of the functions $f \in \mathcal{A}$ for which $\Omega^{\delta}f \in M(\lambda, \phi, t)$. The class $M^{\delta}(\lambda, \phi, t)$ is a special case of the class $M^{g}(\lambda, \phi, t)$ when

$$ g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\delta)}{\Gamma(n+1-\delta)} z^n, \quad (z \in \Delta). $$

Now applying Theorem 2.1 for the function $(f * g)(z) = z + g_2a_2z^2 + g_3a_3z^3 + \cdots$, we get following theorem after an obvious change of the parameter $\mu$:

**Theorem: 3.2** Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ $(g_n > 0)$. If f(z) is given by (1.1) belongs to $M^{g}(\lambda, \phi, t)$ then

$$ \left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{1}{g_3(1+2\lambda)(2+t)(1-t)}\left[B_2 + B_1^2\left(\dfrac{1+t}{1-t}\right) - \mu\dfrac{g_3}{g_2^2}B_1^2\dfrac{(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)}\right] & \text{if } \mu \le \eta_1 \\ \dfrac{B_1}{g_3(1+2\lambda)(2+t)(1-t)} & \text{if } \eta_1 \le \mu \le \eta_2 \\ -\dfrac{1}{g_3(1+2\lambda)(2+t)(1-t)}\left[B_2 + B_1^2\left(\dfrac{1+t}{1-t}\right) - \mu\dfrac{g_3}{g_2^2}B_1^2\dfrac{(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)}\right] & \text{if } \mu \ge \eta_2 \end{cases} $$

where

$$ \eta_1 = \frac{g_2^2(1+\lambda)^2(1-t)}{B_1g_3(1+2\lambda)(2+t)}\left\{-1 + \frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\right\}, $$

$$ \eta_2 = \frac{g_2^2(1+\lambda)^2(1-t)}{B_1g_3(1+2\lambda)(2+t)}\left\{1 + \frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\right\}. $$

The result is sharp.

Since

$$ \Omega^{\delta}f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\delta)}{\Gamma(n+1-\delta)} z^n. $$

We have

$$ g_2 := \frac{\Gamma(3)\Gamma(2-\delta)}{\Gamma(3-\delta)} = \frac{2}{2-\delta}, \tag{3.2} $$

and

$$ g_3 := \frac{\Gamma(4)\Gamma(2-\delta)}{\Gamma(4-\delta)} = \frac{6}{(2-\delta)(3-\delta)}. \tag{3.3} $$

For $g_2$, $g_3$ given by (3.2) and (3.3) respectively, Theorem 3.2 reduces to the following:

**Theorem: 3.3** Let $\delta < 2$. If f(z) is given by (1.1) belongs to $M^{\delta}(\lambda, \phi, t)$ then

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{(2-\delta)(3-\delta)}{6(1+2\lambda)(2+t)(1-t)}\left[B_2+B_1^2\left(\dfrac{1+t}{1-t}\right)-\dfrac{3}{2}\mu\left(\dfrac{2-\delta}{3-\delta}\right)\dfrac{(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)}B_1^2\right] & \text{if } \mu \le \eta_1^* \\ \dfrac{(2-\delta)(3-\delta)B_1}{6(1+2\lambda)(2+t)(1-t)} & \text{if } \eta_1^* \le \mu \le \eta_2^* \\ \dfrac{-(2-\delta)(3-\delta)}{6(1+2\lambda)(2+t)(1-t)}\left[B_2+B_1^2\left(\dfrac{1+t}{1-t}\right)-\dfrac{3}{2}\mu\left(\dfrac{2-\delta}{3-\delta}\right)\dfrac{(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)}B_1^2\right] & \text{if } \mu \ge \eta_2^* \end{cases}$$

where

$$\eta_1^* = \frac{2}{3B_1}\left(\frac{3-\delta}{2-\delta}\right)\left(\frac{(1+\lambda)^2(1-t)}{(1+2\lambda)(2+t)}\right)\left\{-1+\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)\right\},$$

$$\eta_2^* = \frac{2}{3B_1}\left(\frac{3-\delta}{2-\delta}\right)\left(\frac{(1+\lambda)^2(1-t)}{(1+2\lambda)(2+t)}\right)\left\{1+\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)\right\}.$$

The result is sharp.

For the case $\lambda = 0$ in Theorem 3.2, we get Fekete-Szegö inequality for functions to be in the class $S^g(\phi, t)$ which are given by Goyal and Goswami [3].

**Theorem: 3.4** Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ $(g_n > 0)$. If f(z) is given by (1.1) belongs to $M^g(\lambda, \phi, t)$ then

$$\left|a_3-\mu a_2^2\right| \le \frac{B_1}{(1+2\lambda)(2+t)(1-t)g_3}\max\left\{1, \left|\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)-\frac{(1+2\lambda)(2+t)\mu g_3 B_1}{(1+\lambda)^2(1-t)g_2^2}\right|\right\}.$$

The result is sharp.

**Theorem: 3.5** If f(z) is given by (1.1) belongs to $M^\delta(\lambda, \phi, t)$ then

$$\left|a_3-\mu a_2^2\right| \le \frac{B_1(2-\delta)(3-\delta)}{6(1+2\lambda)(2+t)(1-t)}\max\left\{1, \left|\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)-\frac{6(1+2\lambda)(2+t)(2-\delta)\mu B_1}{4(1+\lambda)^2(1-t)(3-\delta)}\right|\right\}.$$

The result is sharp.

Theorems 3.4, Theorem 3.5 were obtained by applying Lemma 1.4.

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