International Journal of Mathematical Archive-2(8), 2011, Page: 1333-1340 Available online through www.ijma.info ISSN 2229-5046

CERTAIN COEFFICIENT INEQUALITIES FOR SAKAGUCHI TYPE FUNCTIONS AND APPLICATIONS TO FRACTIONAL DERIVATIVES

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(Received on: 18-07-11; Accepted on: 02-08-11)


#### Abstract

In the present paper, sharp upper bounds of $\left|\mathrm{a}_{3}-\mu \mathrm{a}_{2}^{2}\right|$ for the functions $f(z)=z+a_{2} z^{2}+a_{3} z^{3}+\cdots$ belonging to a new subclass of Sakaguchi type functions are obtained. Also, application of our results for subclass of functions defined by convolution with a normalized analytic function are given. In particular, Fekete-Szegö inequalities for certain classes of functions defined through fractional derivatives are obtained.


2000 Mathematics Subject Classification: 30C50; 30C45; 30C80.
Keywords: Sakaguchi functions, Analytic functions, Subordination, Fekete-Szegö inequality.

## 1. Introduction

Let $\mathcal{A}$ be the class of analytic functions of the form

$$
\begin{equation*}
\mathrm{f}(\mathrm{z})=\mathrm{z}+\sum_{\mathrm{n}=2}^{\infty} \mathrm{a}_{\mathrm{n}} \mathrm{z}^{\mathrm{n}} \quad(\mathrm{z} \in \Delta:=\{\mathrm{z} \in \mathrm{C} / \mathrm{z} \mid<1\} \tag{1.1}
\end{equation*}
$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. For two functions $\mathrm{f}, \mathrm{g} \in \mathcal{A}$, we say that the function $\mathrm{f}(\mathrm{z})$ is subordinate to $\mathrm{g}(\mathrm{z})$ in $\Delta$ and write $\mathrm{f} \prec \mathrm{g}$ or $\mathrm{f}(\mathrm{z}) \prec \mathrm{g}(\mathrm{z})$, if there exists an analytic function $\mathrm{w}(\mathrm{z})$ with $\mathrm{w}(0)=0$ and $|\mathrm{w}(\mathrm{z})|<1(\mathrm{z} \in \Delta)$, such that $\mathrm{f}(\mathrm{z})=\mathrm{g}(\mathrm{w}(\mathrm{z})),(\mathrm{z} \in \Delta)$. In particular, if the function g is univalent in $\Delta$, the above subordination is equivalent to $f(0)=g(0)$ and $f(\Delta) \subset g(\Delta)$.

A function $\mathrm{f}(\mathrm{z}) \in \mathcal{A}$ is said to be in the class $\mathrm{M}(\lambda, \alpha, \mathrm{t})$ if it satisfies

$$
\begin{gather*}
\operatorname{Re}\left\{\frac{(1-t)\left[\lambda z^{2} f^{\prime \prime}(z)+z f^{\prime}(z)\right]}{(1-\lambda)[f(z)-f(t z)]+\lambda z\left[f^{\prime}(z)-t f^{\prime}(t z)\right]}\right\}>\alpha, \\
|t| \leq 1, t \neq 1,0 \leq \lambda \leq 1,0 \leq \alpha<1 \tag{1.2}
\end{gather*}
$$

For the case $\lambda=0$ in (1.2) we get a Sakaguchi type class $S^{*}(\alpha, t)$. A function $f(z) \in \mathcal{A}$ is said to be in the class $S^{*}(\alpha, t)$ if it satisfies

$$
\begin{equation*}
\operatorname{Re}\left\{\frac{(1-\mathrm{t}) \mathrm{zf} \mathrm{f}^{\prime}(\mathrm{z})}{\mathrm{f}(\mathrm{z})-\mathrm{f}(\mathrm{zt})}\right\}>\alpha, \quad|\mathrm{t}| \leq 1, \mathrm{t} \neq 1 \tag{1.3}
\end{equation*}
$$

which was introduced and studied by Owa et al. [9, 10] .For some $\alpha \in[0,1)$ and for all $z \in \Delta$. For $\lambda=0, \alpha=0$ and $t=-1$ in $M(\lambda, \alpha, t)$, we get the class the class $S^{*}(0,-1)$ studied by Sakaguchi [11]. A function $f(z) \in S^{*}(\alpha,-1)$ is called Sakaguchi function of order $\alpha$.

In this paper, we define the following class $\mathrm{M}(\lambda, \phi, \mathrm{t})$, which is generalization of the class $\mathrm{M}(\lambda, \alpha, \mathrm{t})$.
Definition: 1.1 Let $\phi(z)=1+B_{1} z+B_{2} z^{2}+\cdots$ be univalent starlike function with respect to ' 1 ' which maps the unit disk $\Delta$ onto a region in the right half plane which is symmetric with respect to the real axis, and let $\mathrm{B}_{1}>0$. Then function f $\in \mathcal{A}$ is in the class $\mathrm{M}(\lambda, \phi, \mathrm{t})$ if

$$
\begin{gather*}
\left\{\frac{(1-\mathrm{t})\left[\lambda \mathrm{z}^{2} \mathrm{f}^{\prime \prime}(\mathrm{z})+\mathrm{zf} \mathrm{f}^{\prime}(\mathrm{z})\right]}{(1-\lambda)[\mathrm{f}(\mathrm{z})-\mathrm{f}(\mathrm{tz})]+\lambda \mathrm{z}\left[\mathrm{f}^{\prime}(\mathrm{z})-\mathrm{tf}^{\prime}(\mathrm{tz})\right]}\right\} \prec \phi(\mathrm{z}) \\
|\mathrm{t}| \leq 1, \mathrm{t} \neq 1,0 \leq \lambda \leq 1 \tag{1.4}
\end{gather*}
$$

For $\lambda=0$ in (1.4) we get the class $S^{*}(\phi, t)$ which was defined by Goyal and Goswami [3].
Definition: 1.2 Let $\phi(z)=1+B_{1} z+B_{2} z^{2}+\cdots$ be univalent starlike function with respect to ' 1 ' which maps the unit disk $\Delta$ onto a region in the right half plane which is symmetric with respect to the real axis, and let $\mathrm{B}_{1}>0$. Then function f $\in \mathcal{A}$ is in the class $\mathrm{S}^{*}(\phi, \mathrm{t})$ if

$$
\begin{equation*}
\left\{\frac{(1-\mathrm{t}) \mathrm{zf}^{\prime}(\mathrm{z})}{\mathrm{f}(\mathrm{z})-\mathrm{f}(\mathrm{zt})}\right\} \prec \phi(\mathrm{z}), \quad|\mathrm{t}| \leq 1, \mathrm{t} \neq 1 \tag{1.5}
\end{equation*}
$$

Again $\mathrm{T}(\phi, \mathrm{t})$ denote the subclass of $\mathcal{A}$ consisting functions $\mathrm{f}(\mathrm{z})$ such that $\mathrm{zf}^{\prime}(\mathrm{z}) \in \mathrm{S}^{*}(\phi, \mathrm{t})$.
When $\phi(\mathrm{z})=(1+\mathrm{Az}) /(1+\mathrm{Bz}),(-1 \leq \mathrm{B}<\mathrm{A} \leq 1)$, we denote the subclasses $\mathrm{S}^{*}(\phi, \mathrm{t})$ and $\mathrm{T}(\phi, \mathrm{t})$ by $\mathrm{S}^{*}[\mathrm{~A}, \mathrm{~B}, \mathrm{t}]$ and $\mathcal{T}[\mathrm{A}, \mathrm{B}$, t] respectively.

Obviously $\mathrm{S}^{*}(\phi, 0) \equiv \mathrm{S}^{*}(\phi)$. When $\mathrm{t}=-1$, then $\mathrm{S}^{*}(\phi,-1) \equiv \mathrm{S}_{\mathrm{S}}^{*}(\phi)$, which is a known class studied by Shanmugam et al. [12]. For $t=0$ and $\phi(z)=(1+A z) /(1+B z),(-1 \leq B<A \leq 1)$, the subclass $S^{*}(\phi, t)$ reduces to the class $S^{*}[A, B]$ studied by Janowski [4]. For $0 \leq \alpha<1$ let $S^{*}(\alpha, t):=S^{*}[1-2 \alpha,-1$; t], which is a known class studied by Owa et al. [10] Also,for
$\mathrm{t}=-1$ and $\phi(\mathrm{z})=\frac{1+(1-2 \alpha) \mathrm{Z}}{1-\mathrm{Z}}$, our class reduces to a known class $\mathrm{S}(\alpha,-1)$ studied by Cho et al. ([1], see also [10]).
In the present paper, we obtain the Fekete-Szegö inequality for the functions in the subclass $M(\lambda, \phi, t)$. We also give application of our results to certain functions defined through convolution (or Hadamard product) and in particular, we consider the class $\mathrm{M}^{\delta}(\lambda, \phi, \mathrm{t})$ defined by fractional derivatives.

To prove our main results, we need the following lemma:
Lemma: $\mathbf{1 . 3}$ [6] If $p(z)=1+c_{1} z+c_{2} z^{2}+\cdots$ is an analytic function with positive real part in $\Delta$, then

$$
\left|c_{2}-v_{1}^{2}\right| \leq \begin{cases}-4 v+2 & \text { if } v \leq 0 \\ 2 & \text { if } 0 \leq v \leq 1 \\ 4 v-2 & \text { if } v \geq 1\end{cases}
$$

When $v<0$ or $v>1$, the equality holds if and only if $p(z)$ is $(1+z) /(1-z)$ or one of its rotations. If $0<v<1$, then the equality holds if and only if $p(z)$ is $\left(1+z^{2}\right) /\left(1-z^{2}\right)$ or one of its rotations. If $v=0$, the equality holds if and only if

$$
\mathrm{p}(\mathrm{z})=\left(\frac{1}{2}+\frac{1}{2} \lambda\right) \frac{1+\mathrm{z}}{1-\mathrm{z}}+\left(\frac{1}{2}-\frac{1}{2} \lambda\right) \frac{1+\mathrm{z}}{1-\mathrm{z}} \quad(0 \leq \lambda \leq 1)
$$

or one of its rotations. If $v=1$, the equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\mathrm{v}=0$.
Also the above upper bound is sharp, and it can be improved as follows when $0<\mathrm{v}<1$ :
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$$
\left|\mathrm{c}_{2}-\mathrm{vc}_{1}^{2}\right|+\mathrm{v}\left|\mathrm{c}_{1}\right|^{2} \leq 2 \quad(0<\mathrm{v} \leq 1 / 2)
$$

and

$$
\left|c_{2}-\mathrm{vc}_{1}^{2}\right|+(1-\mathrm{v})\left|\mathrm{c}_{1}\right|^{2} \leq 2 \quad(1 / 2<\mathrm{v} \leq 1) .
$$

Lemma: 1.4 [5] If $p_{1}(z)=1+c_{1} z+c_{2} z^{2}+\cdots$ is a function with positive real part, then

$$
\left|c_{2}-\mu c_{1}^{2}\right| \leq 2 \max \{1,|2 \mu-1|\}
$$

where $\mu$ is complex and the result is sharp for the functions given by

$$
\mathrm{p}(\mathrm{z})=\frac{1+\mathrm{z}^{2}}{1-\mathrm{z}^{2}}, \quad \mathrm{p}(\mathrm{z})=\frac{1+\mathrm{z}}{1-\mathrm{z}}
$$

## 2. MAIN RESULTS

Our main result is contained in the following theorem:
Theorem: 2.1 If $f(z)$ given by (1.1) belongs to $M(\lambda, \phi, t)$, then

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq \begin{cases}\frac{1}{(1+2 \lambda)(2+t)(1-t)}\left[B_{2}+B_{1}^{2}\left(\frac{1+t}{1-t}\right)-\frac{\mu B_{1}^{2}(1+2 \lambda)(2+t)}{(1+\lambda)^{2}(1-t)}\right] & \text { if } \mu \leq \sigma_{1} \\ \frac{B_{1}}{(1+2 \lambda)(2+t)(1-t)} & \text { if } \sigma_{1} \leq \mu \leq \sigma_{2} \\ -\frac{1}{(1+2 \lambda)(2+t)(1-t)}\left[B_{2}+B_{1}^{2}\left(\frac{1+t}{1-t}\right)-\frac{\mu B_{1}^{2}(1+2 \lambda)(2+t)}{(1+\lambda)^{2}(1-t)}\right] & \text { if } \mu \geq \sigma_{2}\end{cases}
$$

where

$$
\sigma_{1}=\frac{(1+\lambda)^{2}(1-t)}{B_{1}(1+2 \lambda)(2+t)}\left\{-1+\frac{B_{2}}{B_{1}}+\frac{B_{1}(1+t)}{(1-t)}\right\}
$$

and

$$
\sigma_{2}=\frac{(1+\lambda)^{2}(1-t)}{B_{1}(1+2 \lambda)(2+t)}\left\{1+\frac{B_{2}}{B_{1}}+\frac{B_{1}(1+t)}{(1-t)}\right\} .
$$

The result is sharp.
Proof: Let $\mathrm{f} \in \mathrm{M}(\lambda, \phi, \mathrm{t})$. Then there exists a Schwarz function $\mathrm{w}(\mathrm{z}) \in \mathcal{A}$ such that

$$
\begin{gather*}
\frac{(1-\mathrm{t})\left[\lambda \mathrm{z}^{2} \mathrm{f}^{\prime \prime}(\mathrm{z})+\mathrm{zf}^{\prime}(\mathrm{z})\right]}{(1-\lambda)[\mathrm{f}(\mathrm{z})-\mathrm{f}(\mathrm{tz})]+\lambda \mathrm{z}\left[\left(\mathrm{f}^{\prime}(\mathrm{z})-\mathrm{tf}(\mathrm{tz})\right]\right.}=\phi(\mathrm{w}(\mathrm{z})) \\
(\mathrm{z} \in \Delta ;|\mathrm{t}| \leq 1, \mathrm{t} \neq 1,0 \leq \lambda \leq 1) \tag{2.1}
\end{gather*}
$$

If $p_{1}(z)$ is analytic and has positive real part in $\Delta$ and $p_{1}(0)=1$, then

$$
\begin{equation*}
\mathrm{p}_{1}(\mathrm{z})=\frac{1+\mathrm{w}(\mathrm{z})}{1-\mathrm{w}(\mathrm{z})}=1+\mathrm{c}_{1} \mathrm{z}+\mathrm{c}_{2} \mathrm{z}^{2}+\cdots \quad(\mathrm{z} \in \Delta) \tag{2.2}
\end{equation*}
$$

From (2.2), we obtain

$$
\begin{equation*}
\mathrm{w}(\mathrm{z})=\frac{\mathrm{c}_{1}}{2} \mathrm{z}+\frac{1}{2}\left(\mathrm{c}_{2}-\frac{\mathrm{c}_{1}^{2}}{2}\right) \mathrm{z}^{2}+\cdots \tag{2.3}
\end{equation*}
$$

Let

$$
\begin{equation*}
p(z)=\frac{(1-t)\left[\lambda z^{2} f^{\prime \prime}(z)+\mathrm{zf}^{\prime}(z)\right]}{(1-\lambda)[f(z)-f(t z)]+\lambda z\left[\left(f^{\prime}(z)-f^{\prime}(t z)\right]\right.}=1+b_{1} z+b_{2} z^{2}+\cdots \quad(z \in \Delta) \tag{2.4}
\end{equation*}
$$

which gives

$$
\begin{align*}
& \mathrm{b}_{1}=(1+\lambda)(1-\mathrm{t}) \mathrm{a}_{2} \quad \text { and } \\
& \mathrm{b}_{2}=(1+\lambda)^{2}\left(\mathrm{t}^{2}-1\right) \mathrm{a}_{2}^{2}+(1+2 \lambda)\left(2-\mathrm{t}-\mathrm{t}^{2}\right) \mathrm{a}_{3} . \tag{2.5}
\end{align*}
$$

Since $\phi(\mathrm{z})$ is univalent and $\mathrm{p} \prec \phi$, therefore using (2.3), we obtain

$$
\begin{equation*}
\mathrm{p}(\mathrm{z})=\phi(\mathrm{w}(\mathrm{z}))=1+\frac{\mathrm{B}_{1} \mathrm{c}_{1}}{2} \mathrm{z}+\left\{\frac{1}{2}\left(\mathrm{c}_{2}-\frac{\mathrm{c}_{1}^{2}}{2}\right) \mathrm{B}_{1}+\frac{1}{4} \mathrm{c}_{1}^{2} \mathrm{~B}_{2}\right\} \mathrm{z}^{2}+\cdots \quad(\mathrm{z} \in \Delta) \tag{2.6}
\end{equation*}
$$

Now from (2.4), (2.5) and (2.6), we have

$$
\begin{aligned}
(1+\lambda)(1-t) a_{2} & =\frac{B_{1} c_{1}}{2}, \\
(1+\lambda)^{2}\left(t^{2}-1\right) a_{2}^{2}+(1+2 \lambda)\left(2-t-t^{2}\right) a_{3}= & \frac{1}{2}\left(c_{2}-\frac{c_{1}^{2}}{2}\right) \mathrm{B}_{1}+\frac{1}{4} c_{1}^{2} B_{2}, \\
& |t| \leq 1, t \neq 1,0 \leq \lambda \leq 1 .
\end{aligned}
$$

Therefore we have

$$
\begin{equation*}
\mathrm{a}_{3}-\mu \mathrm{a}_{2}^{2}=\frac{\mathrm{B}_{1}}{2(1+2 \lambda)(2+\mathrm{t})(1-\mathrm{t})}\left[\mathrm{c}_{2}-\mathrm{vc}_{1}^{2}\right] \tag{2.7}
\end{equation*}
$$

where

$$
\mathrm{v}=\frac{1}{2}\left[1-\frac{\mathrm{B}_{2}}{\mathrm{~B}_{1}}-\mathrm{B}_{1}\left(\frac{1+\mathrm{t}}{1-\mathrm{t}}\right)+\mu \mathrm{B}_{1} \frac{(1+2 \lambda)(2+\mathrm{t})}{(1+\lambda)^{2}(1-\mathrm{t})}\right]
$$

Our result now follows by an application of Lemma 1.3. To shows that these bounds are sharp, we define the functions $\mathrm{K}_{\phi_{\mathrm{n}}}(\mathrm{n}=2,3 \ldots)$ by

$$
\begin{gathered}
\frac{(1-\mathrm{t})\left[\lambda \mathrm{z}^{2} \mathrm{~K}_{\phi_{\mathrm{n}}}^{\prime \prime}(\mathrm{z})+\mathrm{zK}_{\phi_{\mathrm{n}}}^{\prime}(\mathrm{z})\right]}{(1-\lambda)\left[\mathrm{K}_{\phi_{\mathrm{n}}}(\mathrm{z})-\mathrm{K}_{\phi_{\mathrm{n}}}(\mathrm{tz})\right]+\lambda \mathrm{z}\left[\mathrm{~K}_{\phi_{\mathrm{n}}}^{\prime}(\mathrm{z})-\mathrm{tK}_{\phi_{\mathrm{n}}}^{\prime}(\mathrm{tz})\right]}=\phi\left(\mathrm{z}^{\mathrm{n}-1}\right) \\
\mathrm{K}_{\phi_{\mathrm{n}}}(0)=0=\left[\mathrm{K}_{\phi_{\mathrm{n}}}\right]^{\prime}(0)-1
\end{gathered}
$$

and the function $F_{\eta}$ and $G_{\eta}(0 \leq \eta \leq 1)$ by

$$
\begin{gathered}
\frac{(1-t)\left[\lambda z^{2} F_{\eta}^{\prime \prime}(z)+z F_{\eta}^{\prime}(z)\right]}{(1-\lambda)\left[F_{\eta}(z)-F_{\eta}(t z)\right]+\lambda z\left[F_{\eta}^{\prime}(z)-t F_{\eta}^{\prime}(t z)\right]}=\phi\left(\frac{z(z+\eta)}{(1+\eta z)}\right) \\
F_{\mu}(0)=0=\left[F_{\eta}\right]^{\prime}(0)-1
\end{gathered}
$$

and

$$
\begin{gathered}
\frac{(1-\mathrm{t})\left[\lambda \mathrm{z}^{2} \mathrm{G}_{\eta}^{\prime \prime}(\mathrm{z})+\mathrm{zG}_{\eta}^{\prime}(\mathrm{z})\right]}{(1-\lambda)\left[\mathrm{G}_{\eta}(\mathrm{z})-\mathrm{G}_{\eta}(\mathrm{t} \mathrm{z})\right]+\lambda \mathrm{z}\left[\mathrm{G}_{\eta}^{\prime}(\mathrm{z})-\mathrm{tG}_{\eta}^{\prime}(\mathrm{t} \mathrm{z})\right]}=\phi\left(\frac{-\mathrm{z}(\mathrm{z}+\eta)}{(1+\eta \mathrm{z})}\right) \\
\mathrm{G}_{\mu}(0)=0=\left[\mathrm{G}_{\eta}\right]^{\prime}(0)-1
\end{gathered}
$$

Obviously the functions $K_{\phi_{n}}, F_{\eta}, G_{\eta} \in M(\lambda, \alpha, t)$. Also we write $K_{\phi}:=K_{\phi_{2}}$. If $\mu<\sigma_{1}$ or $\mu>\sigma_{2}$, then equality holds if and only if $f$ is $K_{\phi}$ or one of its rotations. When $\sigma_{1}<\mu<\sigma_{2}$, the equality holds if and only if $f$ is $K_{\phi_{2}}$ or one of its
rotations. $\mu=\sigma_{1}$ then equality holds if and only if $f$ is $F_{\eta}$ or one of its rotations. $\mu=\sigma_{2}$ then equality holds if and only if $f$ is $G_{\eta}$ or one of its rotations.

If $\sigma_{1} \leq \mu \leq \sigma_{2}$, in view of Lemma 1.3, Theorem 2.1 can be improved.
Theorem: 2.2 Let $f(z)$ given by (1.1) belongs to $M(\lambda, \alpha, t)$ and $\sigma_{3}$ be given by

$$
\sigma_{3}=\frac{1}{\mathrm{~B}_{1}}\left(\frac{(1+\lambda)^{2}(1-\mathrm{t})}{(1+2 \lambda)(2+\mathrm{t})}\right)\left[\frac{\mathrm{B}_{2}}{\mathrm{~B}_{1}}+\mathrm{B}_{1}\left(\frac{1+\mathrm{t}}{1-\mathrm{t}}\right)\right]
$$

If $\sigma_{1}<\mu \leq \sigma_{3}$, then

$$
\begin{gathered}
\left|a_{3}-\mu a_{2}^{2}\right|+\frac{1}{B_{1}^{2}}\left[\left(B_{1}-B_{2}\right)\left(\frac{(1+\lambda)^{2}(1-t)}{(1+2 \lambda)(2+t)}\right)-B_{1}^{2}\left(\frac{(1+\lambda)^{2}(1+t)}{(1+2 \lambda)(2+t)}\right)+\mu B_{1}^{2}\right]\left|a_{2}\right|^{2} \\
\leq \frac{B_{1}}{(1+2 \lambda)(2+t)(1-t)}
\end{gathered}
$$

If $\sigma_{3}<\mu \leq \sigma_{2}$, then

$$
\begin{gathered}
\left|a_{3}-\mu a_{2}^{2}\right|+\frac{1}{B_{1}^{2}}\left[\left(B_{1}+B_{2}\right)\left(\frac{(1+\lambda)^{2}(1-t)}{(1+2 \lambda)(2+t)}\right)+B_{1}^{2}\left(\frac{(1+\lambda)^{2}(1+t)}{(1+2 \lambda)(2+t)}\right)-\mu B_{1}^{2}\right]\left|a_{2}\right|^{2} \\
\\
\leq \frac{B_{1}}{(1+2 \lambda)(2+t)(1-t)}
\end{gathered}
$$

For $\lambda=0$ in Theorem 2.1 we get Fekete-Szegö inequality for functions to be in the class $S^{*}(\phi, t)$ which was given by Goyal and Goswami [3].

Theorem: 2.3 If $f(z)$ is given by (1.1) belongs to $M(\lambda, \phi, t)$ then

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq \frac{B_{1}}{(1+2 \lambda)(2+t)(1-t)} \max \left\{1,\left|\frac{B_{2}}{B_{1}}+\frac{B_{1}(1+t)}{(1-t)}-\frac{(1+2 \lambda)(2+t)}{(1+\lambda)^{2}(1-t)} \mu B_{1}\right|\right\}
$$

The result is sharp.
Proof: By applying the Lemma 1.4 in (2.7) we get Theorem 2.3. The result is sharp for the functions defined by

$$
\frac{(1-\mathrm{t})\left[\lambda \mathrm{z}^{2} \mathrm{f}^{\prime \prime}(\mathrm{z})+\mathrm{zf} \mathrm{f}^{\prime}(\mathrm{z})\right]}{(1-\lambda)[\mathrm{f}(\mathrm{z})-\mathrm{f}(\mathrm{tz})]+\lambda \mathrm{z}\left[\mathrm{f}^{\prime}(\mathrm{z})-\mathrm{tf}^{\prime}(\mathrm{tz})\right]}=\phi\left(\mathrm{z}^{2}\right)
$$

and

$$
\frac{(1-\mathrm{t})\left[\lambda \mathrm{z}^{2} \mathrm{f}^{\prime \prime}(\mathrm{z})+\mathrm{zf}^{\prime}(\mathrm{z})\right]}{(1-\lambda)[\mathrm{f}(\mathrm{z})-\mathrm{f}(\mathrm{tz})]+\lambda \mathrm{z}\left[\mathrm{f}^{\prime}(\mathrm{z})-\mathrm{tf}(\mathrm{t} \mathrm{z})\right]}=\phi(\mathrm{z}) .
$$

## 3. APPLICATIONS TO FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVES

For two analytic functions $\mathrm{f}(\mathrm{z})=\mathrm{z}+\sum_{\mathrm{n}=0}^{\infty} \mathrm{a}_{\mathrm{n}} \mathrm{z}^{\mathrm{n}}$ and $\mathrm{g}(\mathrm{z})=\mathrm{z}+\sum_{\mathrm{n}=0}^{\infty} \mathrm{g}_{\mathrm{n}} \mathrm{z}^{\mathrm{n}}$, their convolution (or Hadamard product) is defined to be the function $(f * g)(\mathrm{z})=\mathrm{z}+\sum_{\mathrm{n}=0}^{\infty} \mathrm{a}_{\mathrm{n}} \mathrm{g}_{\mathrm{n}} \mathrm{z}^{\mathrm{n}}$. For a fixed $\mathrm{g} \in \mathcal{A}$, let $\mathrm{M}^{\mathrm{g}}(\lambda, \phi, \mathrm{t})$ be the class of functions f $\in \mathscr{A}$ for which $(\mathrm{f} * \mathrm{~g}) \in \mathrm{M}(\lambda, \phi, \mathrm{t})$.

Definition: 3.1 Let $f(z)$ be analytic in a simply connected region of the z-plane containing origin. The fractional derivative of $f$ of order $\delta$ is defined by

$$
\begin{equation*}
{ }_{0} \mathrm{D}_{\mathrm{z}}^{\delta} \mathrm{f}(\mathrm{z}):=\frac{1}{\Gamma(1-\delta)} \frac{\mathrm{d}}{\mathrm{dz}} \int_{0}^{\mathrm{z}}(\mathrm{z}-\zeta)^{-\delta} \mathrm{f}(\zeta) \mathrm{d} \zeta \quad(0 \leq \delta<1), \tag{3.1}
\end{equation*}
$$

where the multiplicity of $(\mathrm{z}-\zeta)^{-\delta}$ is removed by requiring that $\log (\mathrm{z}-\zeta)$ is real for $(z-\zeta)>0$.

Using Definition 3.1, Owa and Srivastava (see [7, 8]; see also [13, 14]) introduced a fractional derivative operator $\Omega^{\delta}$ : $\mathcal{A} \rightarrow \mathcal{A}$ defined by

$$
\left(\Omega^{\delta} \mathrm{f}\right)(\mathrm{z})=\Gamma(2-\delta) \mathrm{z}_{0}^{\delta} \mathrm{D}_{\mathrm{z}}^{\delta} \mathrm{f}(\mathrm{z}), \quad(\delta \neq 2,3,4, \ldots)
$$

The class $\mathrm{M}^{\delta}(\lambda, \phi, \mathrm{t})$ consists of the functions $\mathrm{f} \in \mathcal{A}$ for which $\Omega^{\delta} \mathrm{f} \in \mathrm{M}(\lambda, \phi, \mathrm{t})$. The class $\mathrm{M}^{\delta}(\lambda, \phi, \mathrm{t})$ is a special case of the class $\mathrm{M}^{\mathrm{g}}(\lambda, \phi, \mathrm{t})$ when

$$
\mathrm{g}(\mathrm{z})=\mathrm{z}+\sum_{\mathrm{n}=2}^{\infty} \frac{\Gamma(\mathrm{n}+1) \Gamma(2-\delta)}{\Gamma(\mathrm{n}+1-\delta)} \mathrm{z}^{\mathrm{n}}, \quad(\mathrm{z} \in \Delta)
$$

Now applying Theorem 2.1 for the function $(f * g)(z)=z+g_{2} a_{2} z^{2}+g_{3} a_{3} z^{3}+\cdots$, we get following theorem after an obvious change of the parameter $\mu$ :

Theorem: 3.2 Let $g(z)=z+\sum_{n=2}^{\infty} g_{n} z^{n} \quad\left(g_{n}>0\right)$. If $f(z)$ is given by (1.1) belongs to $M^{g}(\lambda, \phi, t)$ then

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq \begin{cases}\frac{1}{g_{3}(1+2 \lambda)(2+t)(1-t)}\left[B_{2}+B_{1}^{2}\left(\frac{1+t}{1-t}\right)-\mu \frac{g_{3}}{g_{2}^{2}} B_{1}^{2} \frac{(1+2 \lambda)(2+t)}{(1+\lambda)^{2}(1-t)}\right] & \text { if } \mu \leq \eta_{1} \\ \frac{B_{1}}{g_{3}(1+2 \lambda)(2+t)(1-t)} & \text { if } \eta_{1} \leq \mu \leq \eta_{2} \\ -\frac{1}{g_{3}(1+2 \lambda)(2+t)(1-t)}\left[B_{2}+B_{1}^{2}\left(\frac{1+t}{1-t}\right)-\mu \frac{g_{3}}{g_{2}^{2}} B_{1}^{2} \frac{(1+2 \lambda)(2+t)}{(1+\lambda)^{2}(1-t)}\right] & \text { if } \mu \geq \eta_{2}\end{cases}
$$

where

$$
\begin{aligned}
& \eta_{1}=\frac{g_{2}^{2}(1+\lambda)^{2}(1-t)}{B_{1} g_{3}(1+2 \lambda)(2+t)}\left\{-1+\frac{B_{2}}{B_{1}}+B_{1}\left(\frac{1+t}{1-t}\right)\right\}, \\
& \eta_{2}=\frac{g_{2}^{2}(1+\lambda)^{2}(1-t)}{B_{1} g_{3}(1+2 \lambda)(2+t)}\left\{1+\frac{B_{2}}{B_{1}}+B_{1}\left(\frac{1+t}{1-t}\right)\right\} .
\end{aligned}
$$

The result is sharp.
Since

$$
\Omega^{\delta} \mathrm{f}(\mathrm{z})=\mathrm{z}+\sum_{\mathrm{n}=2}^{\infty} \frac{\Gamma(\mathrm{n}+1) \Gamma(2-\delta)}{\Gamma(\mathrm{n}+1-\delta)} \mathrm{z}^{\mathrm{n}}
$$

We have

$$
\begin{equation*}
\mathrm{g}_{2}:=\frac{\Gamma(3) \Gamma(2-\delta)}{\Gamma(3-\delta)}=\frac{2}{2-\delta} \tag{3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{g}_{3}:=\frac{\Gamma(4) \Gamma(2-\delta)}{\Gamma(4-\delta)}=\frac{6}{(2-\delta)(3-\delta)} \tag{3.3}
\end{equation*}
$$

For $g_{2}, g_{3}$ given by (3.2) and (3.3) respectively, Theorem 3.2 reduces to the following:
Theorem: 3.3 Let $\delta<2$. If $f(z)$ is given by (1.1) belongs to $\mathrm{M}^{\delta}(\lambda, \phi, t)$ then

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq \begin{cases}\frac{(2-\delta)(3-\delta)}{6(1+2 \lambda)(2+t)(1-t)}\left[B_{2}+B_{1}^{2}\left(\frac{1+t}{1-t}\right)-\frac{3}{2} \mu\left(\frac{2-\delta}{3-\delta}\right) \frac{(1+2 \lambda)(2+t)}{(1+\lambda)^{2}(1-t)} B_{1}^{2}\right] & \text { if } \mu \leq \eta_{1}^{*} \\ \frac{(2-\delta)(3-\delta) B_{1}}{6(1+2 \lambda)(2+t)(1-t)} & \text { if } \eta_{1}^{*} \leq \mu \leq \eta_{2}^{*} \\ \frac{-(2-\delta)(3-\delta)}{6(1+2 \lambda)(2+t)(1-t)}\left[B_{2}+B_{1}^{2}\left(\frac{1+t}{1-t}\right)-\frac{3}{2} \mu\left(\frac{2-\delta}{3-\delta}\right) \frac{(1+2 \lambda)(2+t)}{(1+\lambda)^{2}(1-t)} B_{1}^{2}\right] & \text { if } \mu \geq \eta_{2}^{*}\end{cases}
$$

where

$$
\begin{aligned}
& \eta_{1}^{*}=\frac{2}{3 \mathrm{~B}_{1}}\left(\frac{3-\delta}{2-\delta}\right)\left(\frac{(1+\lambda)^{2}(1-\mathrm{t})}{(1+2 \lambda)(2+\mathrm{t})}\right)\left\{-1+\frac{\mathrm{B}_{2}}{\mathrm{~B}_{1}}+\mathrm{B}_{1}\left(\frac{1+\mathrm{t}}{1-\mathrm{t}}\right)\right\}, \\
& \eta_{2}^{*}=\frac{2}{3 \mathrm{~B}_{1}}\left(\frac{3-\delta}{2-\delta}\right)\left(\frac{(1+\lambda)^{2}(1-\mathrm{t})}{(1+2 \lambda)(2+\mathrm{t})}\right)\left\{1+\frac{\mathrm{B}_{2}}{\mathrm{~B}_{1}}+\mathrm{B}_{1}\left(\frac{1+\mathrm{t}}{1-\mathrm{t}}\right)\right\} .
\end{aligned}
$$

The result is sharp.
For the case $\lambda=0$ in Theorem 3.2, we get Fekete-Szegö inequality for functions to be in the class $S^{g}(\phi, t)$ which are given by Goyal and Goswami [3].

Theorem: 3.4 Let $g(z)=z+\sum_{n=2}^{\infty} g_{n} z^{n} \quad\left(g_{n}>0\right)$. If $f(z)$ is given by (1.1) belongs to $M^{g}(\lambda, \phi, t)$ then

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq \frac{B_{1}}{(1+2 \lambda)(2+t)(1-t) g_{3}} \max \left\{1,\left|\frac{B_{2}}{B_{1}}+B_{1}\left(\frac{1+t}{1-t}\right)-\frac{(1+2 \lambda)(2+t) \mu g_{3} B_{1}}{(1+\lambda)^{2}(1-t) g_{2}^{2}}\right|\right\}
$$

The result is sharp.
Theorem: 3.5 If $f(z)$ is given by (1.1) belongs to $\mathrm{M}^{\delta}(\lambda, \phi, \mathrm{t})$ then

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq \frac{B_{1}(2-\delta)(3-\delta)}{6(1+2 \lambda)(2+t)(1-t)} \max \left\{1,\left|\frac{B_{2}}{B_{1}}+B_{1}\left(\frac{1+t}{1-t}\right)-\frac{6(1+2 \lambda)(2+t)(2-\delta) \mu B_{1}}{4(1+\lambda)^{2}(1-t)(3-\delta)}\right|\right\} .
$$

The result is sharp.
Theorems 3.4, Theorem 3.5 were obtained by applying Lemma 1.4.

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