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EQUIVALENCE CHROMATIC PARTITION OF A GRAPH

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ABSTRACT

Let $G=(V,E)$ be a simple finite undirected graph. A subset S of V is called an equivalence set if every component of the induced sub graph $\langle S\rangle$ is complete. An equivalence number $\beta_e(G)$ is the maximum cardinality of an equivalence set of G [3]. A proper coloring of a graph is a partition of V(G) into independent sets and the minimum cardinality of such a partition is called the chromatic number of G ($\chi(G)$). A partition of V into equivalence sets is called an equivalence partition of V. The minimum cardinality of an equivalence partition is called the equivalence chromatic number of G and is denoted by $\chi_{eq}(G)$. In this paper, a partition of V(G) into equivalence sets is defined and results are derived.

Keywords: *Equivalence set, Equivalence graph, β_e-excellent, rigid β_e-excellent.*

1. INTRODUCTION

Let $G=(V,E)$ be a simple finite undirected graph. A subset S of V is called an equivalence set if every component of the induced sub graph $\langle S\rangle$ is complete. An equivalence number $\beta_e(G)$ is the maximum cardinality of an equivalence set of G [3]. A proper coloring of a graph is a partition of V(G) into independent sets and the minimum cardinality of such a partition is called the chromatic number of G ($\chi(G)$). Since an equivalence set is a generalization of an independent set, every proper color partition is an equivalence partition. Thus, equivalence partition may be considered as a generalization of proper color partition. A vertex u in V(G) is said to be β_e-good if u belongs to a β_e set of G. G is said to be β_e-excellent [12] if every vertex of G is β_e-good. An equivalence graph is a vertex disjoint union of complete graphs. Several variations of partitions of the vertex set have been studied. Since an equivalence set is a generalization of an independent set, every proper color partition is an equivalence partition. Thus, equivalence partition may be considered as a generalization of proper color partition. The concept of equivalence set, sub chromatic number, generalized coloring and equivalence covering number were studied in [1], [2], [4], [5], [6], [8], [10], [12]. In this paper, a partition of V(G) into equivalence sets is defined and results are derived.

2. EQUIVALENCE CHROMATIC PARTITION

Definition 2.1 [3]: Recall that a subset S of V is an equivalent set if every component of $\langle S\rangle$ is complete.

Definition 2.2: A partition of V into equivalence sets is called an equivalence partition of V. The minimum cardinality of an equivalence partition is called the equivalence chromatic number of G and is denoted by $\chi_{eq}(G)$.

Remark 2.3: Any proper color cross partition is an equivalence partition. Therefore, $\chi_{eq}(G)\le\chi(G)$.

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To find $\chi_{eq}(G)$ for standard graphs

1. $\chi eq(K_n) = 1$
2. $\chi eq(\overline{K}_n) = 1$
3. $\chi eq(K_{1,n}) = 2$
4. $\chi eq(K_{m,n}) = 2$
5. $\chi eq(C_n) = 2$
6. $\chi eq(P_n) = 2$
7. $\chi_{eq}(W_n) = \begin{cases} 2 \text{ if } n = 5 \\ 3 \text{ if } n \geq 6 \end{cases}$
8. $\chi_{eq}(P) = 3$, where P is the Petersen graph.

Proposition 2.4: $\chi_{eq}(G) = 1$ if and only if G is component wise complete.

Proof: Obvious.

Proposition 2.5: $\chi_{eq}(G) = 2$ iff V(G) is the disjoint union of two maximal equivalence sets.

Remark 2.6: If G is a complete k-partite graph then $\chi_{eq}(G) = k$.

Remark 2.7: Given a positive integer k, there exists a graph G such that $\left|\chi_{eq}(G) - \chi(G)\right| = k$.

For example, take $G = K_{k+1}$.

$\chi(G) = k+1$, $\chi_{eq}(G) = 1$.

Definition 2.8 [3]: A dominating set S of G is called an equivalence dominating set if $\langle S \rangle$ is complete. The equivalence dominating number $\gamma_e(G)$ of G is the cardinality of its smallest equivalence dominating set.

Remark 2.9: There is no relationship between $\chi_{eq}(G)$ and $\gamma_e(G)$.

For: $\gamma_e(K_n) = 1$ and $\chi_{eq}(K_n) = 1$

$\gamma_e(P_3) = 1$, $\chi_{eq}(P_3) = 2$

$\gamma_e(\overline{K_n}) = n$, $\chi_{eq}(\overline{K_n}) = 1$

Remark 2.10: Given a positive integer k, there exists a graph G such that $\left|\gamma_e(G) - \chi_{eq}(G)\right| = k$.

Proof: Let G be a (k+2)-partite graph, where each partite sets contains at least two elements. Then $\gamma_e(G) = 2$ (Since the set containing two vertices from different partite sets forms a K_2 and this set is a dominating set) and $\chi_{eq}(G) = k+2$. Therefore $\chi_{eq}(G) - \gamma_e(G) = k$.

Remark 2.11: Let G be a bipartite graph of order ≥ 3 which contains an edge. Then $\gamma_{eq}(G) = 2$. Therefore, for any tree T with at least 3 vertices, $\chi_{eq}(T) = 2$.

Proposition 2.12: Let G be an unicyclic graph. Then $\chi_{eq}(G) = 2$.

Proof: Let G be an unicyclic graph. Let C be the unique cycle in G.
Let V(C) = $\{u_1, u_2, u_3, ..., u_k\}$

Case I: k is even.

Then $\chi_{eq}(C)$ contains the partitions $S_1 = \{u_1,u_3,...,u_{k-1}\}$; $S_2 = \{u_2,u_4,...,u_k\}$.

Let P_i be a path attached at u_i, $1\leq i \leq k$.

Let $V(P_i) = \{u_{i1},u_{i2},...,u_{iri}\}$where r_i may be zero. Let $S_{i1} = \{u_{i1},u_{i3},...,u_{iri}\}$ if r_i is odd and $S_{i1} = \{u_{i1},...,u_{iri-1}\}$if r_i is even. Let $S_{i2} = \{u_{i2},u_{i4},...,u_{iri}\}$ or $\{u_{i2},u_{i4},...,u_{iri-1}\}$if r_i is even or odd. Join S_{i2} with S_1 if $u_i \in S_1$. Otherwise join S_{i2} with S_2. Also join S_{i1} with S_2 if $u_i \in S_1$. Otherwise, join S_{i1} with S_1.The resulting partition$\prod$ of V(G) contains only two elements and hence $\chi_{eq}(G) = 2$.

Case II: k is odd.

Then $\chi_{eq}(C)$ contains the partitions $S_1= \{u_1, u_3,...,u_{k-2}, u_k\}$; $S_2 = \{u_2, u_4,...,u_{k-1}\}$.

Let P_i be a path attached at u_i, $1\leq i \leq k$.

Let $V(P_i) = \{u_{i1}, u_{i2},...,u_{iri}\}$where r_i may be zero. Let $S_{i1} = \{u_{i1}, u_{i3},...,u_{iri}\}$ if r_i is odd and $S_{i1} = \{u_{i1},...,u_{iri-1}\}$if r_i is even. Let $S_{i2} = \{u_{i2}, u_{i4},...,u_{iri}\}$ or $\{u_{i2}, u_{i4},...,u_{iri-1}\}$if r_i is even or odd.

Join S_{i2} with S_1 if $u_i \in S_1$ .Otherwise join S_{i2} with S_2. Also join S_{i1} with S_2 if $u_i \in S_1$. Otherwise join S_{i1} with S_1.The resulting partition $\prod$ of V(G) contains two elements and hence $\chi_{eq}(G) = 2$.

Theorem 2.13: Given any positive integer $k \geq 3$, the problem of deciding $\chi_{eq}(G) \geq k$ is NP-complete for any graph G with $\chi(G) \geq k$.

Proof: Let $\prod = \{V_1, V_2,..., V_r\}$ be a chromatic partition of G where $r = \chi(G)$. Let $V(G) = \{u_1,u_2,...,u_n\}$. Add vertices $u_{11},u_{12},u_{21},u_{22},...,u_{n1},u_{n2}$ to V. Join u_{i1} and u_{i2} with u_i, $1\leq i \leq n$. Also join u_{i1} and u_{i2}, $1\leq i \leq n$. Let H be the resulting graph. Then $\chi_{eq}(H) = r = \chi(G)$. The problem of deciding $\chi(G) \geq k$ is NP-complete if $k \geq 3$. Therefore, the problem of deciding $\chi_{eq}(H) \geq k$ is NP-complete.

Definition 2.14: Let $u \in V(G)$. Let S be a maximum equivalence set of G containing u. $|S|-1$ is called the equivalence degree of u and is denoted by $\deg_{eq}(u)$.

Example 2.15: Let $G = K_{1,n}$.

Let u be the central vertex. Then $\deg_{eq}(u) = 1; \deg_{eq}(v) = n-1$where v is the pendant vertex.

Example 2.16: Let G = K_n. $\deg_{eq}(u) = n-1$ for any vertex u of G.

Example 2.17: Let G= P_5. The vertices 1,2,4,5 have $\deg_{eq}$ as 3 and $\deg_{eq}(3) = 2$.

Example 2.18: Let

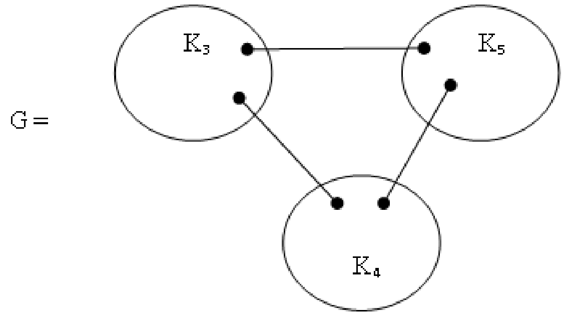

Figure-2.1: Illustration of $\deg_{eq}$

The $\deg_{eq}$ of any vertex of K_3 is 2, $\deg_{eq}$ of any vertex of K_4 is 3 and $\deg_{eq}$ of any vertex of K_5 is 4.

Definition 2.19 [3]: A graph G is said to be an equivalence graph if V(G) is an equivalence set. The maximum cardinality of an equivalence set is denoted by $\beta_e(G)$.

Definition 2.20 [12]: A graph G is said to be a β_e -excellent graph if every vertex of G is contained in a β_e -set of G .

Remark 2.21: Let G be a β_e-excellent graph. Then $\deg_{eq}(u) = \beta_e - 1$ for every vertex u of G. Therefore, G is $\deg_{eq}$ - regular.

Definition 2.22:

$$\Delta_{eq}(G) = \max\{\deg_{eq}(u) : u \in V(G)\}$$
$$\delta_{eq}(G) = \min\{\deg_{eq}(u) : u \in V(G)\}$$

Remark 2.23: $\Delta_{eq}(G) = \beta_e(G) - 1$.

Theorem 2.24: Let G be a graph of order n. Then $\frac{n}{\beta_e(G)} \le \chi_{eq}(G) \le n - \beta_e(G) + 1$.

Proof: Let $\chi_{eq}(G) = k$.

Let $\prod = \{V_1, V_2, ..., V_k\}$ be a χ_{eq} -partition of V(G). Each V_i is an equivalence set and hence $|V_i| \le \beta_e(G)$.

Therefore, $n = |V_1| + |V_2| + ... + |V_k| \le \beta_e(G) + \beta_e(G) + ... + \beta_e(G)_{\text{(k times)}}$.

Therefore, $n \le k\beta_e(G)$.

Therefore, $\frac{n}{\beta_e(G)} \le k = \chi_{eq}(G)$.

Let S be a β_e set of G. Consider the partition $\prod = \{S, \{v_1\}, \{v_2\}, ... \{v_{n-\beta_e(G)}\}\}$ where $v_i \in V - S$, $1 \le i \le n - \beta_e(G)$.

Therefore, $|\prod| \ge \chi_{eq}(G)$.

That is, $n - \beta_e(G) + 1 \ge \chi_{eq}(G)$.

Therefore, $\chi_{eq}(G) \le n - \beta_e(G) + 1$.

Remark 2.25: In the case of K_n, $\beta_e(K_n) = n$, $\chi_{eq}(K_n) = 1$ and $\frac{n}{\beta_e(K_n)} = \frac{n}{n} = 1 = \chi_{eq}(K_n)$. $\chi_{eq}(K_n) = 1 = n - \beta_e(K_n) + 1$.

Problem 2.26: $\chi_{eq}(G) \le 1 + \Delta_{eq}(G)$

It is known that, $\chi_{eq}(G) \le \chi(G) \le 1 + \Delta(G)$.

Problem 2.27: What is the relationship between $\Delta_{eq}(G)$ and $\Delta(G)$?

In the case of star $K_{1,n}$, $\Delta_{eq}(G) = n-1$, $\Delta(G) = n$.

In the case of P_5, $\Delta_{eq} = 3$ and $\Delta = 2$.

In the case of K_n, $\Delta_{eq} = \Delta = n-1$.

Remark 2.28: If G has a full degree vertex, $\Delta(G) = n-1$ and $\Delta_{eq}(G) \le n-1$.

Remark 2.29: $\Delta_{eq}(G) = n-1$ iff $G = K_n$ or $\overline{K_n}$ or G is an equivalence graph.

$\Delta_{eq}(G) \ge \Delta(G)$ if G is an equivalence graph and $\Delta_{eq}(G) = \Delta(G)$ in the case of equivalence graph G iff $G = K_n$

In the case of equivalence graph, $\chi_{eq}(G) = 1$ and $\Delta_{eq}(G) = n-1$.

Remark 2.30: Let $\prod$ be a χ_{eq}-partition of G. Let the number of β_e-sets in the partition be denoted by $\chi_{eq}^{\beta_e}$.

1. In the case of K_n,
   $\chi_{eq} = 1$ and $\chi_{eq}^{\beta_e} = 1$.
2. In the case of P_6,
   $\chi_{eq} = 2$ and $\chi_{eq}^{\beta_e} = 1$.
3. In the case of complete tri-partite graph, where each partite set contains 3 elements,
   $\chi_{eq} = 3$ and $\chi_{eq}^{\beta_e} = 3$.

In general, in the case of complete k-partite graph, where each partite set contains k elements, $\chi_{eq} = k$ and $\chi_{eq}^{\beta_e} = k$.

Remark 2.31: In a χ_{eq}-partition of G, it may happen that none of the sets in the partition is a β_e-set. For example,

In W_7, $\beta_e = 4$, $\Pi = \{\{1,6,7\},\{2,4,5\},\{3\}\}$ is a χ_{eq}- partition and it does not contain any β_e-set.

Remark 2.32: Let S be an equivalence set in G with k components, each component being complete. Then in $\overline{G}$, the vertices of S constitute a complete k-partite graph where each component of S becomes a partite set in $\overline{G}$. Therefore, S becomes an equivalence set in $\overline{G}$ if and only if $\langle S \rangle$ is a complete subgraph of G or a totally disconnected subgraph of G.

REFERENCES

1. M.O. Albertson, R.E. Jamison, S.T. Hedetniemi and S.C.Locke, The subchromatic number of a graph, Discrete Math., 74(1989), 33-49.
2. N. Alon, Covering graphs by the minimum number of equivalence relations, Combinatorica 6 (3) (1986), 201-206.
3. S. Arumugam,M.Sundarakannan, Equivalence Dominating Sets in Graphs, Utilitas Mathematica 91(2013) 231-242.
4. A. Blokhuis and T. Kloks, On the equivalence covering number of split graphs, Information Processing Letters, 54(1995), 301-304.

5. P.Duchet, Representations, noyaux en theorie des graaphes et hyper graphes, These de Doctoral d'EtatI,,Universite Paris VI, 1979.
6. R.D. Dutton and R.C. Brigham, Domination in Claw-Free Graphs, Congr. Numer., 132(1998), 69-75.
7. G.H.Fricke, Teresa, W. Haynes, S.T. Hedetniemi, S.M.Hedetniemi and R.C. Laskar, Excellent trees. preprint.
8. J.Gimbel and C. Hartman, Subcolorings and the subchromatic number of a graph, Discrete Math., 272 (2003), 139-154.
9. Harary, Graph Theory,Addison-Wesley/Narosa,1988.
10. C.Mynhardt and I. Broere, Generalized colorings of graphs, In Y.Alavi, G.Chartrand L.Lesniak,
11. D.R. Lick and C.E. Wall, editors, Graph Theory with Applications to Algorithms and Computer Science, Wiley, (1985),583-594.
12. N.Sridharan and M.Yamuna, A note of Excellent graphs, Ars Combinatoria 78 (2006), pp.267-276.
13. N.Saradha, V.Swaminathan, Excellence with respect to the parameter β_e of a graph, Indian Journal of Applied Research, Vol. 6, issue 10, October 2016, 377-388.
14. M.Yamuna, Excellent-Just Excellent-Very Excellent Graphs, Ph.D. Thesis, Alagappa University, Karaikudi, 2003.

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