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VERY β_e - EXCELLENCE OF A GRAPH

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ABSTRACT

Let $G=(V,E)$ be a simple finite undirected graph. A subset S of V is called an equivalence set if every component of the induced sub graph $\langle S \rangle$ is complete. The equivalence number $\beta_e(G)$ is the maximum cardinality of an equivalence set of G [3]. A vertex u in V(G) is said to be β_e-good if u belongs to a β_e set of G. G is said to be β_e-excellent if every vertex of G is β_e-good. A graph G = (V,E) is said to be very β_e-excellent if there exists a β_e-set S of G such that for every u in V-S, there exists a vertex v in S such that $(S-\{v\})\cup\{u\}$ is β_e-set of G. S is called a very β_e-excellent set of G and G is called a very β_e-excellent graph. An equivalence graph is a vertex disjoint union of complete graphs. The concept of equivalence set, sub chromatic number, generalized coloring and equivalence covering number were studied in [1], [2], [4], [5], [6], [8], [10]. In this paper the concept of very β_e-excellence is studied.

Keywords: *Equivalence set, Equivalence graph, β_e-excellence, Very β_e-excellence.*

1. INTRODUCTION

Gred.H. Fricke et al [7] called a vertex u of a graph G = (V, E) to be μ-good if u is contained in a $\mu(G)$-set of G(where μ is a parameter). G is said to be μ-excellent if every vertex in V is μ-good. A number of results has been proved by taking μ as the domination parameter. Sridharan and Yamuna [12], [13] introduced several types of excellence, one of them being rigid excellence. A graph G is said to be rigid μ-excellent if every vertex of G belongs to a unique μ-set of G. Rigid γ-excellence was studied in [13]. A similar study was made with respect to the parameter β_0 in [11]. A sub set S of V(G) is said to be an equivalence set if every component of $\langle S \rangle$ is complete. A graph G is said to be an equivalence graph if V(G) is an equivalence set. The maximum cardinality of an equivalence set is denoted by $\beta_e(G)$ [3]. In this paper, very β_e-excellence is defined and several results are derived.

2. Very β_e -Excellence of a Graph

Definition 2.1: A graph G = (V,E) is said to be very β_e-excellent if there exists a β_e-set S of G such that for every u in V-S, there exists a vertex v in S such that $(S-\{v\})\cup\{u\}$ is a β_e-set of G. S is called a very β_e-excellent set of G and G is called a very β_e-excellent graph.

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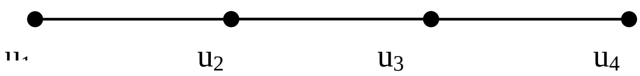
Example 2.2: Consider P_4 with $V(P_4) = \{u_1, u_2, u_3, u_4\}$.



A graph which is very β_e-excellent

Figure-2.1

S={u_1, u_2, u_4} is a β_e-set of P_4. Also P_4 is β_e-excellent. $V - S = \{u_3\}$ and $(S - \{u_2\}) \cup \{u_3\}$ is a β_e set of P_4. Therefore, S is a very β_e-excellent set of P_4 and P_4 is a very β_e-excellent graph.

Remark 2.3: Any very β_e-excellent graph is a β_e-excellent graph.

Proof: Let G be a very β_e-excellent graph and let S be a very β_e-excellent set of G. Let $u \in V - S$. Then there exist $v \in S$ such that $(S - \{v\}) \cup \{u\}$ is a β_e-set of G. Therefore, every vertex of V-S is an element of a β_e-set of G. Since S is a β_e-set of G, every element of V(G) is in a β_e-set of G. Therefore, G is β_e-excellent.

Remark 2.4: A very β_e-excellent graph need not be a rigid β_e-excellent graph. For example, P_4 is a very β_e-excellent graph. But is not a rigid β_e-excellent graph.

Very β_e -excellence for standard graphs

1. K_n is very β_e-excellent for all n.
2. $K_{1,n}$ is not a very β_e-excellent graph for any $n \geq 2$.
3. $\overline{K_n}$ is a very β_e- excellent for all n.
4. W_n is not very β_e-excellent for $n \geq 5$.
5. $K_{m,n}$ is not very β_e-excellent .
6. Petersen graph is not very β_e-excellent.
7. Any equivalence graph is very β_e-excellent.

Proposition 2.5: P_n is very β_e-excellent iff $n = 2,3,4,6,7,9,12$.

Proof: When $n \equiv 2(\mathrm{mod}\,3)$, P_n is not β_e-excellent and hence not very β_e-excellent.
Therefore, the possible values of n are $n = 15r$, $n = 15r+1$, $n = 15r+3$, $n = 15r+4$, $n = 15r+6$, $n = 15r+7$, $n = 15r+9$, $n = 15r+10$, $n = 15r+12$, $n = 15r+13$, $n = 15r+15$ ($r \geq 1$).

Case I: $n = 15r$. Let $n = 3k$. Then $k = 5r$; If $n = 3k$ then $\beta_e = 2k = 10r$.

Since the number of vertices is 15r, there are 3r consecutive five vertices set. For very β_e-excellence, from each set at most 3 vertices can be taken. Therefore, at most 3(3r) =9r vertices can be taken for constructing a very β_e-excellent set. But $\beta_e(P_n)$ =10r where $n = 15r$. Therefore, P_n where $n = 15r$ is not very β_e excellent.

Case II: $n = 15r + 1$.

$n = 3k+1$ implies $k = 5r$; $\beta_e = 2k+1 = 10r+1$.

Since there are 15r+1 vertices, we have 3r five consecutive element sets. From these sets as per the definition of very β_e-excellent set, at most 3 vertices can be taken from each set. The number of possible vertices chosen is 3(3r)+1=9r+1. But $\beta_e = 10r+1$. Therefore, P_n where n = 15r+1 is not very β_e-excellent.

Case III: $n = 15r+3$
n= 3k where $k = 5r+1$, $\beta_e = 2k = 2(5r+1)$.

The number of possible vertices in a very β_e-excellent set chosen is $3(3r)+2 = 9r+2$.

Hence, P_n where n = 15r+3 is not very β_e-excellent.

Case IV: $n = 15r+4$. $n = 3k+1$ where $k = 5r+1, \beta_e = 2k+1 = 2(5r+1)+1 = 10r+3$.

The number of possible vertices chosen with respect to the definition of very β_e-excellent set is $3(3r)+3 = 9r+3$. But $\beta_e = 10r+3$.
Therefore, P_n where n = 15r+4 is not very β_e-excellent.

Case V: $n = 15r+6$, $n = 3k$ where $k = 5r+2$, $\beta_e = 2k = 2(5r+2) = 10r+4$. There are 3r+1 five consecutive elements sets and from each set at most 3 vertices can be chosen is at most $3(3r+1)+1 = 9r+4$. But $\beta_e = 10r+4$. Therefore, P_n where n = 15r+6 is not very β_e-excellent.

Case VI: $n = 15r+7$, $n = 3k+1$ where $k = 5r+2, \beta_e = 2k+1 = 2(5r+2)+1 = 10r+5$.

The number of maximum possible vertices chosen for a very β_e-excellent set is $3(3r+1)+2 = 9r+5$. But $\beta_e = 10r+5$. Therefore, P_n where n = 15r+7 is not very β_e-excellent.

Case VII: $n = 15r+9$; $n = 3k+3$ where $k = 5r+3$, $\beta_e = 2k = 2(5r+3) = 10r+6$.

The number of possible vertices chosen for constructing a very β_e-excellent set is $3(3r+1)+3 = 9r+6$. But $\beta_e = 10r+6$. Therefore, P_n where n = 15r+9 is not very β_e-excellent.

Case VIII: $n = 15r+10$. $n = 3k+1$ where $k = 5r+3$; $\beta_e = 2k+1 = 2(5r+3)+1 = 10r+7$. The number of possible vertices chosen for constructing a very β_e-excellent set is $3(3r+2) = 9r+6$. But $\beta_e = 10r+7$. Therefore, P_n where n= 15r+10 is not very β_e-excellent.

Case IX: $n = 15r+12$, $n = 3k$ where $k = 5r+4$; $\beta_e = 2k = 2(5r+4) = 10r+8$.

The number of possible vertices chosen for constructing a very β_e-excellent set is 3(3r+2)+2=9r+8. But $\beta_e = 10r+8$. Therefore, P_n where $n = 15r+12$ is not very β_e-excellent set.

Case X: $n = 15r+13$. $n = 3k+1$ where $k = 5r+4$; $\beta_e = 2k+1 = 2(5r+4)+1 = 10r+9$.

The number of possible vertices chosen for constructing a very β_e-excellent set is 3(3r+2)+2. But $\beta_e = 10r+9$. Therefore P_n where $n = 15r+13$ is not very β_e-excellent set.

Case XI: $n = 15r + 15$. $n = 3k$ where $k = 5r + 5$; $\beta_e = 2k = 2(5r + 5) = 10r + 10$.

The number of possible vertices chosen for constructing a very β_e-excellent set is 3(3r+5) = 9r+15. But $\beta_e = 10r + 10$. Therefore P_n where $n = 15r + 15$ is not very β_e-excellent set.

When $n = 1,2,3,4$ P_n is clearly very β_e-excellent.

When n =6, $\{u_1,u_2,u_5,u_6\}$is a very β_e-excellent set where $V(P_6) = \{u_1, u_2, u_3, u_4, u_{5,} u_6\}$.

When n =7, $\{u_1,u_2,u_4, u_6, u_7\}$ is a very β_e -excellent set.

When $n = 9$, $\{u_1,u_2,u_4, u_6, u_7, u_9\}$is a very β_e -excellent.

When $n = 10$; $n = 3k + 1$ where $k = 3$. There are two five consecutive elements set in P_{10} and at most 6 element are possible for a very β_e-excellent. Hence P_n is not very β_e-excellent.

When $n = 12$, n = 3k where $k = 4$. $\beta_e(P_n) = 8$.

The set $\{u_1, u_2, u_4, u_6, u_7, u_9, u_{11}, u_{12}\}$ is a very β_e-excellent and hence P_{12} is a very β_e-excellent graph.

When n =13, n = 3k+1 where k=4. $\beta_e(P_{13}) = 9$. There are two five consecutive element sets with 3 elements remaining in the last. Hence at most 6 elements can be taken from the two consecutive elements sets and all the three remaining elements are to be taken for having 9 elements. This might will not give a β_e-set, since 3 consecutive elements cannot be taken in a β_e-set. Hence P_{13} is not very β_e-excellent.

Proposition 2.6: C_n is very β_e-excellent only if n =3,4,5,7,10,13.

Proof: Arguing as in the previous proposition 2.5 the above result is obtained.

Remark 2.7: If a graph G has a unique β_e-set which is not V(G) then G is not very β_e-excellent.

Proposition 2.8: $C_n \circ K_1$ is not very β_e-excellent.

Proof:

Case I: Let n be even.

Let $V(C_n \circ K_1) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$. Any β_e-set S of $C_n \circ K_1$ consists of all v_i's and alternate u_i's. Any vertex outside S cannot come inside by replacing a vertex of S without affecting the equivalence nature of S. Therefore, $C_n \circ K_1$ is not very β_e-excellent.

Case II: Let n be odd.

A similar argument as before shows that there exist no β_e excellent set which is very β_e-excellent.

Observation 2.9: A very β_e-excellent graph may have isolates. Also, there are non-equivalence graphs which have isolates and which are very β_e-excellent.

For example, $K_m \cup \overline{K_n}$ is a very β_e-excellent graph which have isolates, but this is an equivalence graph. $C_4 \cup K_1$ is a non equivalence graph which is very β_e-excellent and which has an isolate.

Remark 2.10: If G is a very β_e-excellent graph then $G \cup \overline{K_m}$ is also very β_e-excellent.

Proposition 2.11: Let G be very β_e-excellent graph without isolates. Let S be a very β_e-excellent set of G. Then for any $u \in S$, $|pn[u,S]| \geq 1$.

Proof: Let G be a very β_e-excellent graph and let S be a very β_e-excellent set of G. Let $u \in S$. Suppose u is an isolate of S and any neighbor of u in G is adjacent with some vertex of S other than u. Then $pn[u,S]=1$. Also, if all the neighbors of u form a complete sub graph with u, then $pn[u,S]=1$.

Corollary 2.12: P_6 is very β_e-excellent. $S=\{u_1,u_2,u_5,u_6\}$ is a very β_e excellent set of G and $pn[u_5,S]=2>1$

Remark 2.13: Let G be a graph without isolates. Let S be a very β_e-excellent set of G. Let $x \in V-S$. Then there exist $u \in S$ such that $(S-\{u\}) \cup \{x\}$ is a β_e-set of G. x need not be a private neighbour of u. For example, P_7 is very β_e-excellent. Let $V(P_7)=\{u_1,u_2,u_3,u_4,u_5,u_6,u_7\}$. Let $S=\{u_1,u_2,u_4,u_6,u_7\}$. Then S is a very β_e-excellent subset of V(G). $(S-\{u_2\}) \cup \{u_3\}$ is a β_e-set of G. But u_3 is not a private neighbour of u_2.

Theorem 2.14: Let G be a graph without isolates. Suppose there exist a β_e-set S of G such that for every $x \in V-S$, there exist $u \in S$ such that $x \in pn(u,S)$. Then G is very β_e-excellent.

Proof: By hypothesis, there exist a β_e-set S of G such that for every $x \in V-S$, there exist $u \in S$ such that $x \in pn(u,S)$. Then $(S-\{u\}) \cup \{x\}$ is a β_e-set of G. Therefore S is a very β_e-excellent set of G. Hence G is a very β_e-excellent graph.

Illustration 2.15: Let $V(C_4)=\{u_1,u_2,u_3,u_4\}$. Let $S=\{u_1,u_2\}$. Then u_3 and u_4 are private neighbours of S.
$(S-\{u_2\}) \cup \{u_3\}$ is a β_e-set.
$(S-\{u_1\}) \cup \{u_4\}$ is a β_e-set.

Theorem 2.16: Let G be a graph such that G is an equivalence graph. Let $V_1,V_2,\ldots,V_k$ be the components of G which are complete. Add vertices $u_1, u_2,\ldots,u_k$. Join u_i only with every vertex of V_i, $1 \leq i \leq k$. Let H be the resulting graph. Then H is very β_e-excellent.

Proof: Clearly H is an equivalence graph and H is very β_e-excellent.

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