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SEPARATION AXIOMS ON TRI STAR TOPOLOGICAL SPACES

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ABSTRACT

In this article, separation axioms in Tri star topological spaces are introduced. Several properties and relationship between the topological spaces induced by separation axioms are analyzed.

Keywords: *T^*_{123}-open, T^*_{123}-pre open, T^*_{123}-T_k spaces T^*_{123}- pre T_k spaces, k = 0, 1, 2, 3.*

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1. INTRODUCTION

Let (X, τ) be a topological space. A set X equipped with two topologies τ_1, τ_2 is called a Bitopological space with topology $\tau = (\tau_1 \cup \tau_2)$. This concept was first introduced by Kelly J.C [6] in 1963. In 1982, Mashhour *et al.* [8] introduced the concept of separation axioms in bitopological spaces. After that many authors worked on separation axioms in bitopological spaces [1], [4]. The study of tri topological spaces was first initiated by Kovar M [7] in 2000. A non empty set X with three topologies τ_1, τ_2, τ_3 is called a tri topological space with topology $\tau' = (\tau_1 \cup \tau_2 \cup \tau_3)$ and is denoted by (X, τ_1, τ_2, τ_3). Hameed and Mohammed Yahya Abid [3] studied separation axioms in tri topological spaces. In 2014, Palaniammal S and Somasundharam S [9] defined another topology $\tau'' = \tau_1 \cap \tau_2 \cap \tau_3$ in tri topological space. In 2016, Stella Irene Mary J and Hemalatha M [10] introduced a new topology called T^*_{123}-topology in tri topological spaces, which is a combination of two bitopologies defined by $\tau = (\tau_1 \cup \tau_3) \cap (\tau_2 \cup \tau_3)$ and studied the various properties of T^*_{123}-open, T^*_{123}-pre open and T^*_{123}-semi open sets. Note that $\tau' \supset T^*_{123} \supset \tau''$.

In this paper, we introduce T^*_{123}-T_k spaces, T^*_{123} –pre T_k spaces, k = 0, 1, 2, 3 based on the separation axioms induced by T^*_{123}-open and T^*_{123}- pre open sets and their properties are analyzed.

2. PRELIMINARIES

Definition 2.1.1: [10] A tri topological space (X, τ_1, τ_2, τ_3) is said to be T^*_{123}-topological space if the topology τ on X is defined by $\tau = (\tau_1 \cup \tau_3) \cap (\tau_2 \cup \tau_3)$, where $(\tau_1 \cup \tau_3)$ and $(\tau_2 \cup \tau_3)$ are bitopologies defined on the bitopological spaces (X, τ_1, τ_3) and (X, τ_2, τ_3) respectively.

Definition 2.1.2: [10] Let (X, τ_1, τ_2, τ_3) be T^*_{123}-topological space and A $\subset$ X is called

1. T^*_{123}-open if A $\subseteq$ $\tau_1 \tau_3$-int($\tau_2 \tau_3$-int A) = $\tau_2 \tau_3$-int($\tau_1 \tau_3$-int A) and T^*_{123}-closed if A $\supseteq$ $\tau_1 \tau_3$-cl ($\tau_2 \tau_3$--cl A).
2. T^*_{123}- pre open if A $\subseteq$ $\tau_1 \tau_3$-int($\tau_2 \tau_3$-cl A) and T^*_{123}- pre closed if $\tau_1 \tau_3$-cl ($\tau_2 \tau_3$-int A) $\subseteq$ A.

The intersection of all T^*_{123}-closed sets (or T^*_{123}-pre closed sets) containing A is called T^*_{123}-closure (or T^*_{123}-pre closure) of A and it is denoted by T^*_{123}-cl(A)(or T^*_{123}- pre cl(A)).

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Definition 2.1.3: [5] Let X be a topological space with topology τ . If Y is a subset of X, the collection τ_Y = {Y $\bigcap$ U | U $\in \tau$ } is a topology on Y, called subspace topology.

Definition 2.1.4: [2] Let (X, τ) be a topological space.

T_0 axiom: If x, y are distinct elements of X, then there exist an open set U $\in \tau$ such that either x $\in$ U and y $\notin$ U or x $\notin$ U and y $\in$ U.

T_1 axiom: If x, y $\in$ X and y $\neq$ x, then there exist two open sets U and V such that U contains x but not y and V contains y but not x.

T_2 axiom: If x, y $\in$ X and y $\neq$ x, then there exist two disjoint open sets U and V containing x and y respectively.

T_3 axiom: If A is a closed subset of X and x be any point of X disjoint from A, then there exist two disjoint open sets U and V containing x and A respectively

3. SEPARATION AXIOMS IN T^*_{123}-SPACE

3.1. T^*_{123}- T_0 space:

Definition 3.1.1: A T^*_{123} topological space (X, τ_1, τ_2, τ_3) is said to be T^*_{123}- T_0 space if and only if for each pair distinct points x, y $\in$ X, there exist a T^*_{123}-open set U such that either x $\in$ U and y $\notin$ U or x $\notin$ U and y $\in$ U.

Example 3.1.2: Let X = {a, b, c} with τ_1 = {X, ϕ }, τ_2 = {X, ϕ ,{a}}, τ_3 = Discrete topology, then for each pair x, y with x $\neq$ y in X, there exist a T^*_{123}-open {x} not containing y.

Theorem 3.1.3: A T^*_{123} topological space (X, τ_1, τ_2, τ_3) is a T^*_{123}- T_0 space if and only if for each pair of distinct points x, y $\in$ X, there exist a subset U of X, which is $\tau_1 \bigcap \tau_2$-open or τ_3-open such that x $\in$ U and y $\notin$ U or x $\notin$ U and y $\in$ U.

Proof: Assume that (X, τ_1, τ_2, τ_3) is a T^*_{123}- T_0 space. By definition 3.1.1, for any two points x, y with x $\neq$ y in X, there exist a T^*_{123}-open set U such that either x $\in$ U and y $\notin$ U or x $\notin$ U and y $\in$ U. Since $\tau = (\tau_1 \bigcup \tau_3) \bigcap (\tau_2 \bigcup \tau_3)$ = ($\tau_1 \bigcap \tau_2$) $\bigcup \tau_3$ and U is T^*_{123}-open, we have U is ($\tau_1 \bigcap \tau_2$)-open or τ_3-open.

Conversely, let x, y in X with x $\neq$ y. By hypothesis, there exist a subset U $\subset$ X, which is ($\tau_1 \bigcap \tau_2$)-open or τ_3-open such that x $\in$ U and y $\notin$ U or x $\notin$ U and y $\in$ U. But τ is union of the ($\tau_1 \bigcap \tau_2$) and τ_3 open sets, implies U is T^*_{123}-open containing x or y. Hence (X, τ_1, τ_2, τ_3) is a T^*_{123}- T_0 space.

Theorem 3.1.4: For a space (X, τ_1, τ_2, τ_3) to be a T^*_{123}- T_0 space, it is sufficient that X with τ_3-as its topology is a T_0-space or X with $\tau_1 \bigcap \tau_2$ - as its topology is a T_0-space.

Proof: Assume that (X, τ_3) is a T_0 space or (X, $\tau_1 \bigcap \tau_2$) is a T_0 space . Then for each pair of distinct points x, y of X, there exist a τ_3-open set U_1 or a $\tau_1 \bigcap \tau_2$- open set U_2 containing either x or y. Since τ = ($\tau_1 \bigcap \tau_2$) $\bigcup \tau_3$, we have U_1 and U_2 are τ -open, and hence U_1 and U_2 are T^*_{123}-open. Consequently, (X, τ_1, τ_2, τ_3) is a T^*_{123}- T_0 space.

Remark 3.1.5: The condition in the above Theorem is not necessary. We show that there exist (X, τ_3) and (X, $\tau_1 \bigcap \tau_2$) spaces which are not T_0, yet (X, τ_1, τ_2, τ_3) is T^*_{123}- T_0 space.

Example 3.1.6: Let X = {a, b, c} with τ_1 = {X, ϕ ,{a},{c},{a, c}}, τ_2 = { X, ϕ ,{a},{b},{a, b}}, τ_3 = {X, ϕ ,{a, c}} then for each pair x, y with x $\neq$ y in X, there exist a subset U of X, such that x $\in$ U and y $\notin$ U or x $\notin$ U and y $\in$ U. Here X with τ_3 and X with $\tau_1 \bigcap \tau_2$-are not a T_0-space, but (X, τ_1, τ_2, τ_3) is a T^*_{123}- T_0 space.

The following Theorem proves a characterization for a T^*_{123}- T_0 space:

Theorem 3.1.7: A T^*_{123} topological space (X, τ_1, τ_2, τ_3) is said to be T^*_{123}- T_0 space if and only if the T^*_{123}-closure of distinct points are distinct.

Proof: Assume that (X, τ_1, τ_2, τ_3) is T^*_{123}- T_0 space and let x, y be two distinct points of X. We show that T^*_{123}-closure of x and y are also distinct. (X, τ_1, τ_2, τ_3) is T^*_{123}- T_0 space implies that there exist a T^*_{123}-open set U such that either $x \in U$ and $y \notin U$. Now U being T^*_{123}-open implies that X-U is T^*_{123}-closed. Also $x \notin X$-U and $y \in X$-U. Since T^*_{123}-cl{y} is the intersection of all T^*_{123}-closed sets containing y, $y \in T^*_{123}$-cl{y} but $x \notin T^*_{123}$-cl{y} as $x \notin X$-U. Similarly $x \in T^*_{123}$-cl{x} but $y \notin T^*_{123}$-cl{x}. Hence T^*_{123}-cl{x} $\neq$ T^*_{123}-cl{y}.

Conversely, Suppose that for any pair of distinct points x, y in (X,τ_1, τ_2, τ_3) we have T^*_{123}-cl{x} $\neq$ T^*_{123}-cl{y}. Then there exist at least one point $z \in X$ such that $z \in T^*_{123}$-cl{x} but $z \notin T^*_{123}$-cl{y}. We claim that $x \notin T^*_{123}$-cl{y}. If $x \in T^*_{123}$-cl{y} then T^*_{123}-cl{x} $\subseteq$ T^*_{123}-cl{y}. So $z \in T^*_{123}$-cl{y} which is contradiction. Hence $x \notin T^*_{123}$-cl{y}. This implies that $x \in (T^*_{123}\text{-cl}\{y\})^c$, which is a T^*_{123}-open set containing x but not y. Hence (X, τ_1, τ_2, τ_3) is a T^*_{123}- T_0 space.

The following Theorem proves the hereditary property of T^*_{123}- T_0 space.

Theorem3.1.8: Every subspace of T^*_{123}- T_0 space is a T^*_{123}- T_0 space in (X, τ_1, τ_2, τ_3).

Proof: Let (X, τ_1, τ_2, τ_3) be a T^*_{123}- T_0 space and (Y, $\tau_1^{'}, \tau_2^{'}, \tau_3^{'}$) be its subspace where $\tau_1^{'}\tau_3^{'} \cap \tau_2^{'}\tau_3^{'}$ is subspace topology of $\tau_1\tau_3 \cap \tau_2\tau_3$ on Y. Let y_1, y_2 be any two distinct point of Y and hence of X. Now as (X, τ_1, τ_2, τ_3) is a T^*_{123}- T_0 space there exist a T^*_{123}-open set U such that $y_1 \in U$ and $y_2 \notin U$. Then $U \cap Y$ is a T^*_{123}-open in (Y, $\tau_1^{'}, \tau_2^{'}, \tau_3^{'}$), which contains y_1 but does not contain y_2. Hence Y is T^*_{123}- T_0 space.

3.2. T^*_{123}- T_1 space:

Definition 3.2.1: A T^*_{123} topological space (X, τ_1, τ_2, τ_3) is said to be T^*_{123}- T_1 space if and only if for any given pair of distinct points x and y, there exist two T^*_{123}-open sets U and V such that U contains x but not y and V contains y but not x.

Remark 3.2.2: Every T^*_{123}- T_1 space is T^*_{123}- T_o space, but converse not true.

Example 3.2.3: Consider the set N, of all natural numbers and let $\tau_1 = \tau_2 = \{N, \phi\}$ and τ_3 be the collection consisting of N, ϕ and those subsets of N of the form {1,2,3,…n}, $n \in N$. Clearly the space (N, τ_1, τ_2, τ_3) is a T^*_{123}- T_o space. But it is not a T^*_{123}- T_1 space, because if we consider two distinct points m and n (m < n)and if we choose U = {1,2,3,…m} then $m \in U$ and $n \notin U$, but there does not exist any T^*_{123}-open set V contains n but not m.

Theorem 3.2.4: A T^*_{123} topological space (X, τ_1, τ_2, τ_3) is a T^*_{123}- T_1 space iff for each pair of distinct points x, y $\in$ X, there exist subsets U, V of X, which are $\tau_1 \cap \tau_2$ -open or τ_3 -open such that $x \in U$ and $y \notin U$ or $x \notin V$ and $y \in V$.

Proof: Assume that (X, τ_1, τ_2, τ_3) is a T^*_{123}- T_1 space. By definition 3.2.1, for any two points x,y with $x \neq y$ in X, there exist T^*_{123}-open sets U and V such that U contains x but not y and V contains y but not x. Since $\tau = (\tau_1 \cap \tau_2) \cup \tau_3$ and U, V $\in \tau$, implies U is $\tau_1 \cap \tau_2$ -open or τ_3 -open.

Conversely, let x, y in X with $x \neq y$. By hypothesis, there exist subsets U, $V \subset X$, which are ($\tau_1 \cap \tau_2$)-open or τ_3 -open such that $x \in U$ and $y \notin U$ or $x \notin V$ and $y \in V$. Then U and V must belongs to the collection τ. This implies U is T^*_{123}-open containing either x or y. Hence (X, τ_1, τ_2, τ_3) is a T^*_{123}- T_1 space.

Theorem 3.2.5: A space $(X, \tau_1, \tau_2, \tau_3)$ to be T^*_{123}- T_1 space, it is sufficient that X with τ_3 -is a T_1-space or X with $\tau_1 \cap \tau_2$ -is a T_1-space.

Proof: Assume that (X, τ_3) is a T_1 space or $(X, \tau_1 \cap \tau_2)$ is a T_1 space. Then for each pair of distinct points x, y of X, there exist a τ_3 -open sets U_1 and V_1, such that U_1 contains x but not y and V_1 contains y but not x or $\tau_1 \cap \tau_2$ -open sets U_2 and V_2 such that U_2 contains x but not y and V_2 contains y but not x. Since $\tau = (\tau_1 \cap \tau_2) \cup \tau_3$, we have U_1, V_1 and U_2, V_2 are T^*_{123}-open sets. Hence $(X, \tau_1, \tau_2, \tau_3)$ is a T^*_{123}- T_1 space.

Remark 3.2.6: The condition in the above theorem is not necessary. We show that there exist (X, τ_3) and $(X, \tau_1 \cap \tau_2)$ spaces which are not T_1, yet $(X, \tau_1, \tau_2, \tau_3)$ is T^*_{123}- T_1 space.

Example 3.2.7: Let X = {a, b, c} with τ_1 = {X, ϕ, {a}, {c}, {b, c}, {a, c}}, τ_2 = { X, ϕ, {a}, {b, c}}, τ_3 = {X, ϕ, {a, b}, {c}} then for every pair of distinct points of X there exists T^*_{123}-open sets U and V contains each points respectively. Here X with τ_3 and X with $\tau_1 \cap \tau_2$ -are notT_1-space, but $(X, \tau_1, \tau_2, \tau_3)$ is a T^*_{123}- T_1 space.

Theorem 3.2.8: A T^*_{123} Topological space $(X, \tau_1, \tau_2, \tau_3)$ is a T^*_{123}- T_1 space if and only if every singleton subset {x} of X is a T^*_{123}- closed set.

Proof: Let X be a T^*_{123}- T_1 space and x be arbitrary point of X. If $y \in \{x\}^c$, then $y \neq x$. Since X is T^*_{123}- T_1 space and $y \neq x$ there must exist an T^*_{123}-open set U_y such that $y \in U_y$ but not x. Thus for each $y \in \{x\}^c$, there exist an T^*_{123}-open set U_y such that $y \in U_y \subseteq \{x\}^c$. Therefore $\{x\}^c = \bigcup \{ y \mid y \neq x\} \subseteq \bigcup \{U_y \mid y \neq x \} \subseteq \{x\}^c$ and so $\{x\}^c = \bigcup \{U_y \mid y \neq x \}$. Since U_y is T^*_{123}-open set and the arbitrary union of T^*_{123}-open sets is T^*_{123}-open, we have $\{x\}^c$ is T^*_{123}-open. Hence {x} is T^*_{123}-closed.

Conversely, let x and y are two distinct points of X such that {x} and {y} are T^*_{123}-closed set. Then $\{x\}^c$ and $\{y\}^c$ are T^*_{123}-open sets in X such that $y \in \{x\}^c$ but $x \notin \{x\}^c$ and $x \in \{y\}^c$ but $y \notin \{y\}^c$. Hence X is T^*_{123}- T_1 space.

Remark 3.2.9: A topological space $(X, \tau_1, \tau_2, \tau_3)$ is a T^*_{123}- T_1 space if and only if every finite subset of X is T^*_{123}-closed.

Definition 3.2.10: Let X be a T^*_{123}-topological space and $A \subset X$, $x \in X$ is a limit point of A if for every T^*_{123}-open set B containing x, $B - \{x\} \cap A \neq \phi$.

Theorem 3.2.11: If $(X, \tau_1, \tau_2, \tau_3)$ be a T^*_{123}- T_1 space, then the following statements are equivalent:

i) $x \in X$ is a T^*_{123}-limit point of A where $A \subset X$.
ii) Every T^*_{123}-open set containing x, contains infinite number of point of A.

Proof:
(i) $\Rightarrow$ (ii): Assuming that $x \in X$ is a T^*_{123}-limit point of A where $A \subset X$ and U is any T^*_{123}-open set containing x, we shall show that U contains infinitely many points of A. Suppose U contains only finite number of points of A other than x. Let $V = U - \{x\} \cap A$, then V being a finite subset of T^*_{123}- T_1 space, is closed and hence V^c is T^*_{123}-open. Take $W = U \cap V^c$, which is also T^*_{123}-open and $x \in W$ implies W does not contain a point of A other than x. This means x is not a T^*_{123}-limit point of A which is contradiction to our assumption. Hence U must contain infinite number of points of A other than x.

(ii) $\Rightarrow$ (i): It is obviously true by the definition of T^*_{123}-limit point.

3.3. T^*_{123}- T_2 space:

Definition 3.3.1: A T^*_{123} topological space $(X, \tau_1, \tau_2, \tau_3)$ is said to be T^*_{123}- T_2 space if and only if for every pair of distinct points x, y of X, there exist disjoint T^*_{123}-open sets U and V containing x and y respectively.

Remark 3.3.2: Every T^*_{123}- T_2 space is a T^*_{123}- T_1 space, but converse need not be true.

Theorem 3.3.3: Every discrete space is T^*_{123}- T_2 space while no indiscrete space consisting of at least two points is T^*_{123}- T_2 space.

Proof: We know that every singleton set is T^*_{123}-open in a discrete space, therefore every pair of distinct points of a discrete space will have disjoint T^*_{123}-open sets and hence every discrete space is T^*_{123}- T_2 space. On the other hand an indiscrete space consisting of at least two points is not a T^*_{123}- T_2 space since the whole space is the only T^*_{123}- open set of each point, so that any two distinct points cannot have disjoint T^*_{123}- open sets.

3.4. T^*_{123}- T_3 space:

Definition 3.4.1: A T^*_{123} topological space $(X, \tau_1, \tau_2, \tau_3)$ is said to be T^*_{123}-regular if for each pair consisting of a point x and a T^*_{123}-closed set B disjoint from x, there exist disjoint T^*_{123}- open sets containing x and B respectively. A T^*_{123} –topological space is said to be T^*_{123}-T_3 space if it is T^*_{123}- regular and its points are T^*_{123}- closed.

Theorem 3.4.2: A T^*_{123} topological space $(X, \tau_1, \tau_2, \tau_3)$ is said to be T^*_{123}- T_3 space iff for each pair consisting of a point x and a T^*_{123}-closed set B disjoint from x, there exist disjoint subsets U, V of X, which are $\tau_1 \cap \tau_2$-open or τ_3-open containing x and B respectively.

Proof: Assume that $(X, \tau_1, \tau_2, \tau_3)$ is a T^*_{123}- T_3 space. By definition 3.4.1, for each pair consisting of a point x and a T^*_{123}-closed set B disjoint from x, there exist disjoint T^*_{123}- open sets U and V containing x and B respectively and each of its points are T^*_{123}- closed. Since $\tau = (\tau_1 \cap \tau_2) \cup \tau_3$ and U, V$\in \tau$, implies U and V are in $\tau_1 \cap \tau_2$-open or τ_3-open containing x and B respectively.
Conversely, let x, y in X with $x \neq y$. By hypothesis, each pair consisting of a point x and a T^*_{123}-closed set B disjoint from x, there exist disjoint subsets U, V of X, which are $\tau_1 \cap \tau_2$-open or τ_3-open containing x and B respectively. Since $\tau = (\tau_1 \cap \tau_2) \cup \tau_3$, then U and V must belongs to the collection τ. This implies U and V are T^*_{123}-open sets containing x and B respectively. Hence $(X, \tau_1, \tau_2, \tau_3)$ is a T^*_{123}- T_1 space.

Theorem 3.4.3: A space $(X, \tau_1, \tau_2, \tau_3)$ to be T^*_{123}- T_3 space, it is sufficient that X with τ_3-is a T_3-space or X with $\tau_1 \cap \tau_2$ -is a T_3-space.

Proof: Assume that (X, τ_3) is a T_3 space or $(X, \tau_1 \cap \tau_2)$ is a T_3 space . Then for each pair consisting of a point x and a T^*_{123}-closed set B disjoint from x, there exists disjoint τ_3- open sets U_1 and V_1, such that U_1 contains x but not B and V_1 contains B but not x or $\tau_1 \cap \tau_2$-open sets U_2 and V_2 such that U_2 contains x but not B and V_2 contains B but not x. Since $\tau = (\tau_1 \cap \tau_2) \cup \tau_3$, we have U_1,V_1 and U_2,V_2 are T^*_{123}-open sets. Hence $(X, \tau_1, \tau_2, \tau_3)$ is a T^*_{123}- T_3 space.

Remark 3.4.4: The condition in the above theorem is not necessary. We show that there exist (X, τ_3) and $(X, \tau_1 \cap \tau_2)$ spaces which are not T_3, yet $(X, \tau_1, \tau_2, \tau_3)$ is T^*_{123}- T_3 space.

Example 3.4.5: Let X = {a, b, c} with τ_1 = {X, ϕ,{c},{b, c},{a, c}}, τ_2 = { X, ϕ,{a},{b, c}}, τ_3 = {X, ϕ,{a}} . This space X is T^*_{123}- T_3 space. Here X with τ_3 and X with $\tau_1 \cap \tau_2$ -are not T^*_{123}-T_3-space, but $(X, \tau_1, \tau_2, \tau_3)$ is a T^*_{123}-T_3 space.

3.5. T^*_{123}- pre T_0 space:

Definition 3.5.1: A T^*_{123} topological space $(X, \tau_1, \tau_2, \tau_3)$ is said to be T^*_{123}-pre T_0 space if and only if for any pair of distinct points x, y $\in$ X, there exist a T^*_{123}- pre open set, which contains one of them but not the other.

Theorem 3.5.2: If {x} is T^*_{123}- pre open for some $x \in X$, then $x \notin T^*_{123}$-pre cl{y} for all $x \neq y$.

Proof: Let {x} be T^*_{123}- pre open set for some $x \in X$, then X-{x} is T^*_{123}- pre closed. Also $x \notin$ X-{x}. If $x \in T^*_{123}$-pre cl{y} for some $y \neq x$, then x, y both are in all the closed sets containing y, so $x \in$ X-{x} which is contradiction. Hence $x \notin T^*_{123}$-pre cl{y}

Theorem 3.5.3: If every non trivial subset of a tri topological space (X, τ_1, τ_2, τ_3) is either T^*_{123}-pre open or T^*_{123}-pre closed then number of T^*_{123}-pre open and T^*_{123}-pre closed sets are equal.

Proof: Suppose if number of T^*_{123}-pre open and T^*_{123}-pre closed sets are not equal, then number of T^*_{123}-pre open sets may be greater than T^*_{123}-pre closed sets or number of T^*_{123}-pre closed sets may be greater than number of T^*_{123}-pre open sets. Without loss of generality we assume that, number of T^*_{123}-pre closed sets are higher than number of T^*_{123}-pre open sets. We know that the only T^*_{123}-pre closed sets are complement of T^*_{123}-pre open sets, implies number of T^*_{123}-pre open sets must be equal to T^*_{123}-pre closed sets. From this we can find some of T^*_{123}-pre open sets are also T^*_{123}-pre closed, which is contradiction to our hypothesis. Hence number of T^*_{123}-pre open and T^*_{123}-pre closed sets are equal.

Theorem 3.5.4: If every non trivial subset of a tri topological space (X, τ_1, τ_2, τ_3) is either T^*_{123}-pre open or T^*_{123}-pre closed then (X, τ_1, τ_2, τ_3) is T^*_{123}-pre T_0 space.

Proof: Assume that every non trivial subset of a tri topological space (X, τ_1, τ_2, τ_3) is either T^*_{123}-pre open or T^*_{123}-pre closed. Suppose (X, τ_1, τ_2, τ_3) is not T^*_{123}-pre T_0 space, then for atleast one of distinct points x, y of X, there does not exist a T^*_{123}-pre open set contains either x or y. This implies both x and y are in the same T^*_{123}-pre open set or same T^*_{123}-pre closed set, means {x} and {y} does not belongs to collections T^*_{123}-pre open sets and T^*_{123}-pre closed sets. But by our hypothesis, every non trivial subset of a tri topological space (X, τ_1, τ_2, τ_3) is either T^*_{123}-pre open or T^*_{123}-pre closed set. So our assumption, (X, τ_1, τ_2, τ_3) is not T^*_{123}-pre T_0 space, is wrong.

Theorem 3.5.5: Every T^*_{123}-T_0 space is T^*_{123}-pre T_0 space.

Proof: Let (X, τ_1, τ_2, τ_3) be a T^*_{123}-T_0 space, then for every pair of distinct points x, y of X, there exist a T^*_{123}-open set contains either x or y. We know that every T^*_{123}-open set is T^*_{123}-pre open, this implies every pair of distinct points x, y of X, there exist a T^*_{123}-pre open set contains either x or y. Hence (X, τ_1, τ_2, τ_3) is a T^*_{123}-pre T_0 space.

Theorem 3.5.6: In any T^*_{123}- topological space (X, τ_1, τ_2, τ_3), if distinct points have distinct T^*_{123}-pre closure then X is T^*_{123}-pre T_0 space.

Proof: Let x, y$\in$ X, with $x \neq y$ and also T^*_{123}-pre cl{x} is not equal to T^*_{123}-pre cl{y}. Hence there exist z $\in$ X such that z$\in$ T^*_{123}-pre cl{x} but z $\notin$ T^*_{123}-pre cl{y}or z$\in$ T^*_{123}-pre cl{y} but z $\notin$ T^*_{123}-pre cl{x}. Now without loss of generality, let z$\in$ T^*_{123}-pre cl{x} but z $\notin$ T^*_{123}-pre cl{y}. We claim that x $\notin$ T^*_{123}-pre cl{y}. If x$\in$ T^*_{123}-pre cl{y}, then T^*_{123}-pre cl{x} is contained in T^*_{123}-pre cl{y}. Hence z $\in T^*_{123}$-pre cl{y}, which is contradiction. This means that x $\notin$ T^*_{123}-pre cl{y}, hence x $\in$ T^*_{123}-pre $cl\{y\}^c$, which is a T^*_{123}-pre open set containing x but not y. Then (X, τ_1, τ_2, τ_3) is a T^*_{123}-pre T_0 space.

3.6. T^*_{123}- pre T_1 space:

Definition 3.6.1: A T^*_{123} topological space (X, τ_1, τ_2, τ_3) is said to be T^*_{123}- pre T_1 space if and only if for any given pair of distinct points x and y, there exist two T^*_{123}-pre open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Remark 3.6.2: Every T^*_{123}- pre T_1 space is T^*_{123}- pre T_0 space, but converse need not be true.

Example 3.6.3: Let X = {a, b, c, d} with τ_1= { X, ϕ }, τ_2 = {X, ϕ,{d},{c},{c, d}}, τ_3 = {X, ϕ,{a}, {a, b, c}, {a, c}, {a, c, d},{a, d}}. Here T^*_{123}- pre open sets are X, ϕ,{a}, {a, b, c},{a, c},{a, c, d},{a, d}. This is T^*_{123}- pre T_0 space, but not T^*_{123}- pre T_1 space since for b and c cannot have distinct T^*_{123}- pre open sets.

Theorem 3.6.4: A T^*_{123} Topological space (X, τ_1, τ_2, τ_3) is a T^*_{123}- pre T_1 space if and only if every singleton subset {x} of X is a T^*_{123}- pre closed set.

Proof: Let X be a T^*_{123}- pre T_1 space and x be arbitrary point of X. If $y \in \{x\}^c$, then $y \neq x$. Since X is T^*_{123}- pre T_1 space and $y \neq x$ there must exist an T^*_{123}-pre open set U_y such that $y \in U_y$ but not x. Thus for each $y \in \{x\}^c$, there exist an T^*_{123}-pre open set U_y such that $y \in U_y \subseteq \{x\}^c$. Therefore $\bigcup \{ y \mid y \neq x\} \in \bigcup \{U_y \mid y \neq x \} \subseteq \{x\}^c$ and so

$\{x\}^c = \bigcup \{U_y \mid y \neq x\}$. Since U_y is T^*_{123}-pre open set and the arbitrary union of T^*_{123}-pre open set is T^*_{123}-pre open and so $\{x\}^c$. Hence {x} is T^*_{123}-pre closed set.
Conversely, let x and y be two distinct points of X such that {x} and {y} are T^*_{123}-pre closed sets. Then $\{x\}^c$ and $\{y\}^c$ are T^*_{123}-pre open sets in X such that $y \in \{x\}^c$ but $x \notin \{x\}^c$ and $x \in \{y\}^c$ but $y \notin \{y\}^c$. Hence X is T^*_{123}- pre T_1 space.

Theorem 3.6.5: Every finite T^*_{123}- pre T_1 space is discrete.

Proof: Let X be a finite T^*_{123}- pre T_1 space. $A \subset X$ be any arbitrary is finite set. By Theorem 3.6.4, every {x} in $(X, \tau_1, \tau_2, \tau_3)$ is T^*_{123}- pre closed for all $x \in X$.
Consequently $A = \bigcup \{\{x\} \mid x \in X\}$ = a finite union of T^*_{123}- pre closed sets and hence A is T^*_{123}- pre closed. Since X-A is also finite, $X\text{-}A = \bigcup \{\{x\} \mid x \in A^c\}$ = a finite union of T^*_{123}- pre closed sets. This implies X-A is a T^*_{123}- pre closed set, and A is T^*_{123}- pre open. Hence $(X, \tau_1, \tau_2, \tau_3)$ is a discrete space.

3.7. T^*_{123}- pre T_2 space, T^*_{123}- pre-irreducible:

Definition 3.7.1: A T^*_{123} topological space $(X, \tau_1, \tau_2, \tau_3)$ is said to be T^*_{123}- pre T_2 space if and only if for every pair of distinct points x, y of X, there exist disjoint T^*_{123}-pre open sets U and V containing x and y respectively. Also every T^*_{123}- pre T_2 space is T^*_{123}- pre T_1 space.

Theorem 3.7.2: Every singleton subset of T^*_{123}- pre T_2 space is T^*_{123}- pre closed.

Proof: Suppose that $(X, \tau_1, \tau_2, \tau_3)$ is T^*_{123}- pre T_2 space and x, y are distinct points of X. Since the space is T^*_{123}- pre T_2 space there exist a T^*_{123}- pre open set U of y such that $x \notin U$. Hence y cannot be a limit point of {x} and the derived set of {x} is empty. This implies $\{\bar{x}\} = \{x\}$. Hence {x} is closed.

Theorem 3.7.3: If $(X, \tau_1, \tau_2, \tau_3)$ be a T^*_{123} – topological space, then the following statements are equivalent:
(i) $\tau_1 \tau_3 \cap \tau_2 \tau_3$ is T^*_{123}- pre T_2 topology for X.
(ii) The intersection of all T^*_{123}- pre closed sets of each point of X is a singleton.

Proof:
(i) $\Leftrightarrow$ (ii): Let $\tau_1 \tau_3 \cap \tau_2 \tau_3$ be a T^*_{123}- pre T_2 topology for X and x, y be distinct points of X. Then there exist T^*_{123}- pre open sets M_1 and M_2 such that $x \in M_1$, $y \in M_2$ and $M_1 \cap M_2 = \phi$. Moreover $x \in X\text{-}M_2$, is T^*_{123}- pre closed and $y \notin X\text{-}M_2$, implies y does not belongs to intersection of all T^*_{123}- pre closed sets of x. Hence the intersection of all T^*_{123}- pre closed sets of x is the singleton {x}, since y is arbitrary.

Conversely, if $\{x\} \in X$ is the intersection of all the T^*_{123}- pre closed sets then any $y \in X$ and $y \neq x$ implies y does not not belong to intersection of all T^*_{123}- pre closed sets of x. Then there exist a T^*_{123}- pre closed set N of x such that $y \notin N$, implies there exist a T^*_{123}- pre open set O such that $x \in O \subset N$. This implies O and X-N are T^*_{123}- pre open sets, such that $x \in O$, $y \in X\text{-}N$ and $O \cap X\text{-}N = \phi$. Then the space $(X, \tau_1, \tau_2, \tau_3)$ is T^*_{123}- pre T_2 space.

Definition 3.7.4: A T^*_{123} topological space is said to be T^*_{123} – pre irreducible, if it cannot be expressed as the union of two proper T^*_{123} – pre closed subsets of X.

Theorem 3.7.5: T^*_{123}- pre closure of every one point set is T^*_{123}- pre irreducible.

Proof: Let $A \subset X$ be a T^*_{123}- pre closure of $x \in X$. Suppose $A = A_1 \bigcup A_2$ where A_1 and A_2 are proper T^*_{123}- pre closed subsets of A. But one of these must contain x, which is contradiction to the fact that A is the smallest T^*_{123}- pre closed set containing x. Hence T^*_{123}- pre closure of every one point set is T^*_{123}- pre irreducible.

Theorem 3.7.6: In a T^*_{123}- pre T_2 space the only T^*_{123}- pre irreducible subsets are one point sets.

Proof: Let $(X, \tau_1, \tau_2, \tau_3)$ be T^*_{123}-pre T_2 space and A be T^*_{123}- pre irreducible subset of X. Then A cannot be expressed as the union of two proper T^*_{123} – pre closed subsets of X. Suppose A has more than one element then A can be written as union of singleton sets. By Theorem 3.7.2, every singleton subset of T^*_{123}- pre T_2 space is T^*_{123}- pre closed. Hence A is union of proper T^*_{123}- pre closed sets, which is contradiction to T^*_{123}- pre irreducibility of A. Hence the only T^*_{123}- pre irreducible subsets of T^*_{123}- pre T_2 spaces are one point sets.

3.8. T^*_{123}- pre regular, T^*_{123}- pre T_3 space:

Definition 3.8.1: A T^*_{123} topological space $(X, \tau_1, \tau_2, \tau_3)$ is said to be T^*_{123}-pre regular if for each pair consisting of a point x and a T^*_{123}- pre closed set B disjoint from x, there exist disjoint T^*_{123}- pre open sets containing x and B respectively.

Definition 3.8.2: A T^*_{123} –topological space is said to be T^*_{123}-pre T_3 space if it is T^*_{123}- pre regular and singleton sets are T^*_{123}- pre closed.

Definition 3.8.3: In a T^*_{123} topological space $(X, \tau_1, \tau_2, \tau_3)$, a T^*_{123} –pre neighborhood of a point (or a set) in X is an T^*_{123} –pre open set which contains the point (or the set).

Theorem 3.8.4:
(i) Every T^*_{123}-pre regular T_1 space is a T^*_{123}-pre T_3 space.
(ii) A T^*_{123}-topological space $(X, \tau_1, \tau_2, \tau_3)$ is T^*_{123}-pre T_3 space, then $(X, \tau_1, \tau_2, \tau_3)$ is T^*_{123}-pre T_1 space

Proof:

(i) Let $(X, \tau_1, \tau_2, \tau_3)$ be a T^*_{123}-pre T_1 space and x, y be two distinct points. Then since X is a T^*_{123}-pre T_1 space, {x} is a T^*_{123}-pre closed set, also $y \notin \{x\}$. Hence by definition of T^*_{123}-pre regular, there exist disjoint open sets G and H such that $\{x\} \in G$ and $y \in H$. Hence every T^*_{123}-pre regular T_1 space is a T^*_{123}-pre T_3 space.

(ii) Assume that $(X, \tau_1, \tau_2, \tau_3)$ be a T^*_{123}-pre T_3 space. This implies that given a T^*_{123}- pre closed $F \subset X$ and $x \in X$ such that $x \notin F$, there exist T^*_{123} – pre open sets G, $H \subset X$ such that $x \in G$, $F \subset H$ and $G \bigcap H = \phi$. Any arbitrary $y \in F$, $x \notin F$ implies $y \neq x$. Therefore for distinct x, $y \in X$, there exist T^*_{123} –open sets G, $H \subset X$ such that $x \in G$, $y \in H$ and $G \bigcap H = \phi$. Hence $(X, \tau_1, \tau_2, \tau_3)$ is a T^*_{123}-pre T_2 space. By definition 3.7.1, every T^*_{123}-pre T_2 space is T^*_{123}-pre T_1 space. Hence a T^*_{123}-pre T_3 space is a T^*_{123}-pre T_1 space.

Theorem 3.8.5: A T^*_{123}-topological space $(X, \tau_1, \tau_2, \tau_3)$ is T^*_{123}-pre T_3 space iff each $x \in X$, there exist a T^*_{123}-pre neighborhood of x which contains the closure of another T^*_{123} –pre neighborhood of x.

Proof: Assume that $(X, \tau_1, \tau_2, \tau_3)$ be a T^*_{123}-pre T_3 space. Then for a T^*_{123}- pre closed $F \subset X$ and $x \in X$ such that $x \notin F$, there exist T^*_{123} – pre open sets G, $H \subset X$ such that $x \in G$, $F \subset H$ and $G \bigcap H = \phi$. $G \bigcap H = \phi \Rightarrow G \subset X - H \Rightarrow \overline{G} \subset \overline{X - H}$ = X-H. Since X-H is T^*_{123}- pre closed implies $\overline{G} \subset$ X-H$\subset$ X-F. And F is T^*_{123}- pre closed implies X-F is T^*_{123}- pre open. Therefore given a T^*_{123}- neighborhood X-F of $x \in X$, there exist a T^*_{123}- neighborhood G of x such that $x \in G \subset \overline{G} \subset X - F$.

Conversely, if each T^*_{123} –neighborhood of $x \in X$ contains the closure of another T^*_{123}- neighborhood of x. Consider a T^*_{123}- pre closed set $F \subset X$ and any $x \in X$ disjoint from F. Then F is T^*_{123}- pre closed and $x \notin F \Rightarrow x \in X$-F is T^*_{123}-pre open, this means X-F is T^*_{123}-neighborhood of $x \in X$. By assumption there exist a T^*_{123} –neighborhood G of x, such that $x \in G \subset \overline{G} \subset X - F$. Now $G \bigcap \left(X - \overline{G}\right) = \phi$ and $x \in \overline{G} \subset X - F$ implies, $F \subset X - \overline{G}$, also $X - \overline{G}$ is T^*_{123}- pre open. Conclusively, given a T^*_{123}- pre closed set F and any $x \in X$ such that $x \notin F$, there exist T^*_{123}-pre open sets G and X-$\overline{G}$ such that $x \in G$, $F \subset X - \overline{G}$ and $G \bigcap \left(X - \overline{G}\right) = \phi$ implies $(X, \tau_1, \tau_2, \tau_3)$ be a T^*_{123}-pre regular space. Hence X is T^*_{123}-pre T_3 space.

Theorem 3.8.6: The product of T^*_{123}-pre T_3 spaces is a T^*_{123}-pre T_3 space.

Proof: Let $\{X_\alpha\}$ be a family of T^*_{123}-pre T_3 space and $X = \prod X_\alpha$. Let $x = (x_\alpha)$ be a point of X and U be a T^*_{123}- pre open set containing x in X. Choose a T^*_{123}-pre open $\prod U_\alpha$ about x contained in U. Choose for each α, a T^*_{123}-pre open set V_α of x_α in X_α, such that $\overline{V_\alpha} \subset U_\alpha$. Take $V = \prod V_\alpha$, then $\overline{V} = \prod \overline{V_\alpha}$. By Theorem 3.8.5, $\prod \overline{V_\alpha} \subset \prod U_\alpha \subset U$. It follows that $\overline{V} \subset \prod U_\alpha \subset U$, so that X is a T^*_{123}-pre T_3 space.

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