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ONE LINE PROOF OF LEMOINE'S PEDAL TRIANGLE THEOREM

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ABSTRACT

In this article we present a new and shortest proof of Lemoine's Pedal Triangle Theorem, in our present proof we use the properties of Maneeals and the metric relation of centroid.

Keywords: *Maneeals, Lemoine's Pedal Triangle Theorem, Lemoine point.*

1. INTRODUCTION

Lemoine's Pedal Triangle Theorem: *Lemoine's pedal triangle theorem states that the Lemoine (symmedian) point of a triangle is the unique point which is the centroid of its own pedal triangle (see figure-1).*

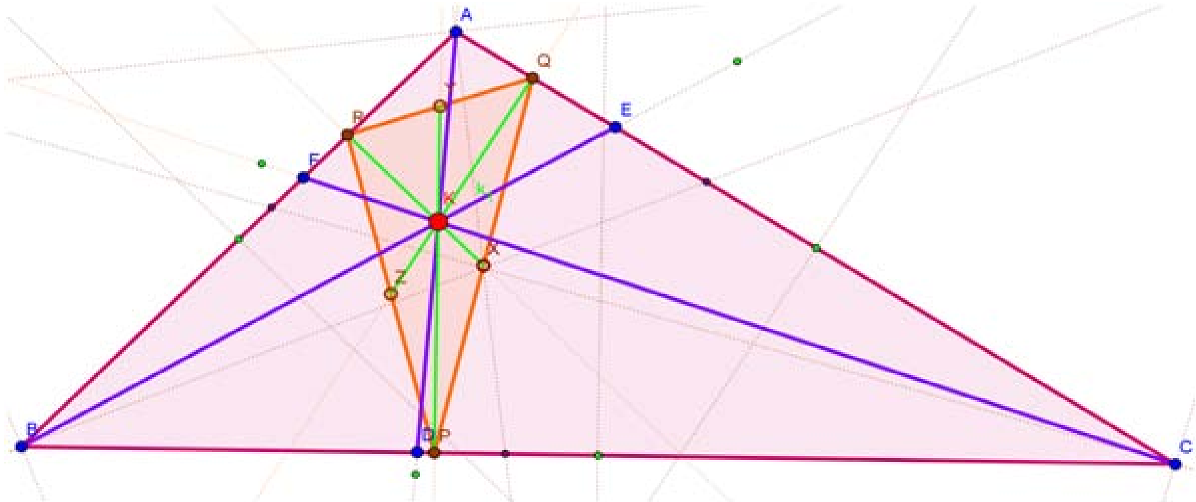

Figure -1: From figure -1 it is clear the symmedian point K of ΔABC is the centroid of ΔPQR

The proof of this theorem can be found in Honsberger [10, p.72]. Darij Grinberg [6] and Cosmin Pohoata[3] has given a synthetic proofs for this theorem. The uniqueness part was conjectured by Clark Kimberling [2], In the article [5, §2.22] it was proved that among all the Maneeals, Maneeal point of order 2 (which is a symmedian point) only exhibits such property. And it was confirmed that it is true for all other points in the plane of the triangle with their works by Barry Wolk [1], Jean-Pierre Ehrmann [7], and Paul Yiu [9, §4.6.2].

In this article we try to prove this theorem using the idea of metric relation of Centroid which is shorter than the existing proofs.

2. NOTATION AND BACKGROUND

Let ABC be a non equilateral triangle. We denote its side-lengths by a, b, c, its area by Δ and its classical center centroid as G. Let us call the cevians AD, BE and CF as symmedians (which are the reflections of medians with respect to the angular bisectors and they divide the opposite sides in the ratio of squares of other two sides) and their point of intersection K as Lemoine point. Let P, Q and R are the vertices of pedal triangle of Lemoine point (K).

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We note some of the metric properties of Lemoine point (K) discussed in [4] and in [5] (by replacing n as 2 in the metric properties of Maneeals since Maneeal point of order 2 is Lemoine point)

(a). The ratio in each symmedian divides the opposite side is given by $\frac{BD}{DC}=\frac{c^2}{b^2}$, $\frac{CE}{EA}=\frac{a^2}{c^2}$ and $\frac{AF}{FB}=\frac{b^2}{a^2}$

(b).The ratio in which K divides each symmedian is given by $\frac{AK}{KD}=\frac{b^2+c^2}{a^2}$, $\frac{BK}{KE}=\frac{a^2+c^2}{b^2}$ and $\frac{CK}{KF}=\frac{a^2+b^2}{c^2}$

(c). The Perpendicular distance from K to the sides BC, CA and AB is given by

$$KP=\frac{2\Delta a}{a^2+b^2+c^2},\quad KQ=\frac{2\Delta b}{a^2+b^2+c^2}\ \text{and}\ KR=\frac{2\Delta c}{a^2+b^2+c^2}$$

Hence it is clear that $KP^2+KQ^2+KR^2=\frac{4\Delta^2}{a^2+b^2+c^2}$

(d). The sides of Pedal triangle of K is given by

$$PQ=\frac{2\Delta}{a^2+b^2+c^2}\sqrt{2a^2+2b^2-c^2},$$

$$QR=\frac{2\Delta}{a^2+b^2+c^2}\sqrt{2c^2+2b^2-a^2}\ \text{and}\ RP=\frac{2\Delta}{a^2+b^2+c^2}\sqrt{2c^2+2a^2-b^2}$$

Hence it is clear that $PQ^2+QR^2+RP^2=\frac{12\Delta^2}{a^2+b^2+c^2}=3\left(KP^2+KQ^2+KR^2\right)$

Now we state a preposition which gives the metric relation of centroid G

3. PREPOSITION

The sum of the squares of the distances of the vertices of a triangle from any point is equal to the sum of the squares of their distances from the centroid increased by three times the square of the distance between the point and the centroid.

That is., if G is the centroid of triangle ABC and M be any point in the plane of the triangle then

$$3GM^2=AM^2+BM^2+CM^2-\left(AG^2+BG^2+CG^2\right)$$

$$=AM^2+BM^2+CM^2-\frac{1}{3}\left(AB^2+BC^2+CA^2\right) \tag{1}$$

The proof of this preposition is available in [8] and (1) can also be proved using a theorem [5, §2.12] related to the metric relation of Maneeal point of order n (since centrod is also maneeal point of order 0). Actually the preposition 1 is a particular case of a more general theorem proved in Robert Simson's *Apollonii Pergaei Locorum Planorum Libri II., pp. 179 – 180 (1729).*

Now we prove our main theorem.

4. MAIN THEOREM

Lemoine's Pedal Triangle Theorem: *The Centroid (G^1) of ΔPQR is acts as Lemoine point (K) of ΔABC.* (Ω)

Proof: To prove **(Ω),** it is enough to prove that $G^1K^2=0$

So from preposition-1, by replacing M as K it is enough to prove that

$$3\left(KP^2+KQ^2+KR^2\right)=PQ^2+QR^2+RP^2$$

It is a straightforward true statement using (c), (d)

Which completes the proof of Lemoine's Pedal Triangle Theorem.

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6. REFERENCES

1. *B. Wolk, Hyacinthos message 19, December 27, 1999.*
2. *C. Kimberling, Hyacinthos message 1, December 22, 1999.*
3. *Cosmin Pohoata, "A short proof of Lemoine's Theorem." Forum Geometricorum 8 (2008): 97—98.*
4. *Dasari Naga Vijay Krishna, Weitzenbock inequality – 2 proofs in a more geometrical way using the idea of "lemoine point" and "Fermat point", GeoGebra International Journal of Romania (GGIJRO), volume-4, No.1, 2015, page no.89, 90.*
5. *Dasari Naga Vijay Krishna, Jakub Kabat, "Several observations about Maneeals - a peculiar system of lines", Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica XV (2016), Ann. Univ. Paedagog. Crac. Stud. Math. 15 (2016), 51-68.*
6. *D. Grinberg, New Proof of the Symmedian Point to be the centroid of its pedal triangle, and the Converse, available at http://de.geocities.com/darij grinberg.*
7. *J.-P. Ehrmann, Hyacinthos message 95, January 8, 2000.*
8. *J.S. Mackay, The triangle and its scribed circles, Proceedings of Edinburg Mathematical Society, volume -1, 1883.*
9. *P. Yiu, Introduction to the Geometry of the Triangle, Florida Atlantic University Lecture Notes, 2001.*
10. *R. Honsberger, Episodes of 19th and 20th Century Euclidean Geometry, Math. Assoc. America, 1995.*

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