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On πgb- Closed Sets in Topological Spaces

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Abstract

In this paper a new class of sets called πgb-closed set is introduced and its properties are studied. Further the notion of πgb-$T_{1/2}$ space and πgb-continuity are introduced.

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Keywords: *πgb -closed, πgb -open, πgb - continuous, πgb -$T_{1/2}$ spaces.*

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1. Introduction

Andrijevic [3] introduced a new class of generalized open sets in a topological space, the so-called b-open sets. This type of sets was discussed by Ekici and Caldas [11] under the name of γ-open sets. The class of b-open sets is contained in the class of semi-pre-open sets and contains all semi-open sets and preopen sets. The class of b-open sets generates the same topology as the class of preopen sets. Since the advent of these notions, several research paper with interesting results in different respects came to existence ([1, 3, 6, 11, 12, 21, 22, 23]). Levine [16] introduced the concept of generalized closed sets in topological space and a class of topological spaces called $T_{½}$ spaces. Extensive research on generalizing closedness was done in recent years as the notions of a generalized closed, generalized semi-closed, α-generalized closed, generalized semi-pre-open closed sets were investigated in [2,7,16,18,19].The finite union of regular open sets is said to be π-open. The complement of a π-open set is said to be π-closed.

The aim of this paper is to study the notion of πgb-closed sets and its various characterizations are given in this paper. In Section 3, we study basic properties of πgb-closed sets. In Section 4, we characterize πgb-open sets**.** Finally in section 5, πgb-continuous and πgb-irresolute functions are discussed.

2. Preliminaries

Throughout this paper (X, τ) and (Y, τ) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ) cl(A) and int(A) denote the closure of A and the interior of A respectively.(X, τ) will be replaced by X if there is no chance of confusion.

Let us recall the following definitions which we shall require later.

Definition 2.1: A subset A of a space (X, τ) is called

(1) a preopen set [17] if A⊂ int (cl (A)) and a preclosed set if cl (int (A))⊂ A;

(2) a semi-open set[15] if A⊂ cl(int(A)) and a semi-closed set if int (cl(A))⊂A;

(3) a α -open set[20] if A⊂int (cl(int (A))) and a α -closed set if cl (int(cl (A)))⊂ A;

(4) a semi-preopen set[1] if A⊂ cl (intcl(A)) and a semi-pre-closed set if int (cl (int(A))) ⊂ A;

(5) a regular open set if A= int (cl(A)) and a regular closed set if A= cl(int (A));

(6) b-open [3] or sp-open [8], γ –open [11] if A ⊂ cl(int(A))∪int (cl(A)).

The complement of a b-open set is said to be b-closed [3]. The intersection of all b-closed sets of X containing A is called the b-closure of A and is denoted by bCl(A). The union of all b-open sets of X contained in A is called b-interior of A and is denoted by bInt(A). The family of all b-open (resp. α-open, semi-open,preopen, β-open,, b-closed, preclosed) subsets of a space X is denoted by bO(X)(resp. αO(X), SO(X), PO(X), βO(X), bC(X), PC(X)) and the

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collection of all b-open subsets of X containing a fixed point x is denoted by bO(X, x).

The sets SO(X, x), αO(X, x), PO(X, x), βO(X, x) are defined analogously.

Lemma 2.2 [3]: Let A be a subset of a space X. Then

(1) bCl(A) = sCl(A) ∩ pCl(A) = A∪[Int(Cl(A)) ∩Cl(Int(A))];
(2) bInt(A) = sInt(A) ∪ pInt(A) = A∩ [Int(Cl(A)) ∪ Cl(Int(A))];

Definition 2.3: A subset A of a space (X, τ) is called

(1) a generalized closed (briefly g-closed)[16] if cl(A) ⊂ U whenever A ⊂ U and U is open.
(2) a generalized b-closed (briefly gb-closed)[13] if bcl(A) ⊂ U whenever A ⊂ U and U is open.
(3) πg-closed [10] if cl(A) ⊂ U whenever A⊂ U and U is π-open.
(4) πgp-closed [24] if pcl(A)⊂U whenever A⊂U and U is π-open.
(5) πgα-closed [14] if αcl(A) ⊂U whenever A⊂U and U is π-open.
(6) πgsp-closed [25] if spcl(A) ⊂U whenever A⊂U and U is π-open.
(7) πgs-closed [4] if scl(A) ⊂U whenever A⊂U and U is π-open.

Definition 2.4: A function f: (X, τ) → (Y, σ) is called

(1) π- irresolute [4] if $f^{-1}(V)$ is π- closed in (X, τ) for every π-closed of (Y, σ);
(2) b-irresolute: [11] if for each b-open set V in Y,$f^{-1}(V)$ is b-open in X;
(3) b-continuous: [11] if for each open set V in Y,$f^{-1}(V)$ is b-open in X.

3. πgb-closed sets

Definition 3.1: A subset A of (X,τ) is called πgb -closed if bcl(A) ⊂ U whenever A ⊂U and U is π-open in (X, τ). By πGBC(τ) we mean the family of all πgb- closed subsets of the space(X, τ).

Theorem 3.2:

1. Every closed set is πgb-closed
2. Every g-closed is πgb-closed
3. Every α-closed set is πgb-closed
4. Every pre-closed set is πgb-closed
5. Every gb-closed set is πgb-closed
6. Every πg -closed set is πgb-closed.
7. Every πgp -closed set is πgb-closed
8. Every πgα-closed set is πgb-closed
9. Every πgs-closed set is πgb-closed.
10. Every πgb-closed set is πgsp-closed.

Proof: Straight forward Converse of the above need not be true as seen in the following examples.

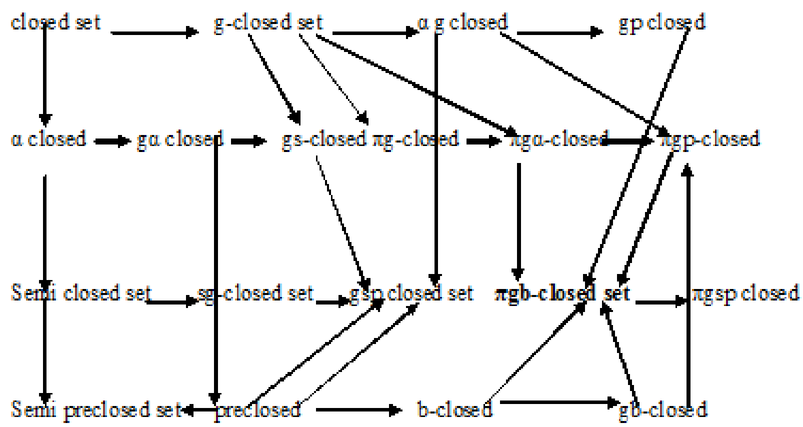
Example3.3: Consider X={a, b, c, d}, τ = {Φ,{a},{d},{a, d},{c, d},{a, c, d},X}.Let A={c}. Then A is πgb-closed but not closed, g-closed, α-closed, pre-closed, gb-closed, πg-closed.

Example 3.4: Let X = {a, b, c, d, e} and τ = { Ø,{a, b},{c, d],{a, b, c, d},X}.Let A={a}. Therefore A is πgb-closed but not πgα- closed.

Example3.5: Let X = {a, b ,c, d, e} and τ = {Ø,{a, b},{c, d],{a, b, c, d},X}.

Let A = {a, b}. Therefore A is πgb -closed but not πgp- closed.

Remark 3.6: The above discussions are summarized in the following diagram.



Theorem 3.7: If A is π-open and πgb- closed, then A is b-closed.

Proof: Let A is π-open and πgb-closed. Let A⊂A where A is π-open. Since A is πgb- closed, bcl (A) ⊂A. Then A = bcl(A).Hence A is b-closed**.**

Theorem 3.8: Let A be πgb-closed in(X, τ).Then bcl(A)-A does not contain any non empty π-closed set.

Proof: Let F be a non empty π-closed set such that F⊂ bcl(A)-A. Since A is πgb-closed, A ⊂ X-F where X-F is π-open implies bcl (A) ⊂ X-F. Hence F⊂ X-bcl(A).Now, F⊂ bcl (A)∩(X-bcl (A)) implies F=Φ which is a contradiction. Therefore bcl (A) does not contain any non empty π- closed set.

Corollary 3.9: Let A be πgb -closed in(X, τ).Then A is b-closed iff bcl(A)-A is π closed.

Proof: Let A be b-closed. Then bcl(A) = A. This implies bcl(A)-A= Φ which is π- closed. Assume bcl(A)-A is π-closed. Then bcl (A)-A= Φ. Hence, bcl (A) = A

Remark 3.10: Finite union of πgb-closed sets need not be πgb-closed.

Example 3.11: Consider X= {a, b, c}, τ ={ Φ,{a},{b},{a, b},X}. Let A = {a}, B = {b}.Here A and B are πgb- closed but A ∪ B = {a, b} is not πgb-closed.

Remark 3.12: Finite intersection of πgb -closed sets need not be πgb- closed.

Example3.13: Consider X = {a, b, c, d}, τ = {Φ,{a},{b},{a, b},{a, b, c}.{a, b, d},X}. Let A = {a, b, c}, B = {a, b. d}.Here A and B are πgb-closed but A ∩ B = {a, b} is not πgb-closed.

Definition 3.14[5]: Let (X, τ) be a topological space A⊂ X and x∈ X is said to be b-limit point of A iff every b-open set containing x contains a point of A different from x.

Definition 3.15[5]: Let (X, τ) be a topological space, A⊂ X. The set of all b-limit points of A is said to be b-derived set of A and is denoted by D_b[A]

Lemma 3.16[5]: If D (A) = D_b (A), then we have cl(A) = bcl(A)

Lemma 3.17[5]: If D (A) ⊂D_b (A) for every subset A of X. Then for any subsets F and B of X, we have bcl(F∪B) = bcl(F)∪bcl(B)

Theorem 3.18: Let A and B be πgb- closed sets in(X, τ) such that D[A]⊂D_b[A] and D[B] ⊂ D_b[B].Then A∪B is πgb-closed

Proof: Let U be π-open set such that A∪B⊂U. Since A and B are πgb-closed sets we have bcl(A)⊂U and bcl(B)⊂U.Since D[A]⊂D_b[A] and D[B]⊂D_b[B],by lemma 3.16,cl(A) =bcl(A) and cl(B) = bcl(B). Thus bcl(A∪B)⊂cl(A∪B)=cl(A)∪cl(B)= bcl(A)∪ bcl(B)⊂U .This implies A∪B is πgb-closed.

Theorem 3.19: If A is πgb-closed set and B is any set such that A ⊂B⊂ bcl (A), then B is πgb-closed set.

Proof: Let B ⊂ U and U be π-open. Given A ⊂ B. Then A ⊂ U
Since A is πgb- closed, A⊂ U implies bcl(A) ⊂ U. By assumption it follows that
b cl(B) ⊂ b cl (A) ⊂ U. Hence B is a πgb-closed set.

4. πgb- open sets

Definition 4.1: A set A ⊂ X is called πgb-open if and only if its complement is πgb- closed.

Remark 4.2: bcl (X-A) = X-bint (A)

By πGBO(τ) we mean the family of all πgb-open subsets of the space(X, τ).

Theorem 4.3: If A ⊂ X is πgb-open iff F ⊂ bint(A) whenever F is π-closed and F⊂ A

Proof: Necessity: Let A be πgb-open. Let F be π-closed and F ⊂ A. Then X-A ⊂ X-F where X-F is π-open. By assumption, bcl(X-A) ⊂ X-F. By remark 4.2, X-bint (A)⊂ X-F. Thus F ⊂ b int (A).

Sufficiency: Suppose F is π- closed and F ⊂ A such that F⊂ bint(A).Let X-A ⊂ U where U is π-open. Then X-U ⊂ A where X-U is π-closed. By hypothesis, X-U⊂bint (A). X-bint(A) ⊂ U. bcl (X-A) ⊂ U. Thus X-A is πgb-closed and A is πgb-open.

Theorem 4.4: If bint (A) ⊂ B ⊂ A and A is πgb-open then B is πgb-open.

Proof: Let bint (A) ⊂B⊂A. Thus X-A⊂X-B ⊂ bcl (X-A) .Since X-A is πgb- closed, by theorem 3.19, (X-A) ⊂ (X-B) ⊂ b cl (X-A) implies (X-B) is πgb- closed.

Remark 4.5: For any A ⊂ X, bint (bcl(A)-A)= Φ

Theorem 4.6: If A ⊂ X is πgb- closed, then bcl(A)-A is πgb- open.

Proof: Let A be πgb- closed let F be π-closed set.F ⊂ b cl (A)-A.By theorem 3.8, F= Φ. By remark4.5, bint (bcl A-A) = Φ.Thus F ⊂ bint (bcl(A)-A).Thus bcl (A)-A is πgb- open.

Lemma 4.7[24]: Let A ⊂X. If A is open or dense, then πO(A, τ /A)=V∩A such that V ∈ πO(X, τ).

Theorem 4.8: Let B ⊂ A ⊂ X where A is πgb-closed and π-open set. Then B is πgb- closed relative to A iff B is πgb-closed in X.

Proof: Let B⊂A⊂X , where A is πgb-closed and π-open set. Let B be πgb-closed in A. Let B⊂U where U is π-open in X.Since B⊂A, B = B∩A⊂U∩A, this implies bcl(B) = bcl_A(B)⊂U∩A⊂U. Hence, B is πgb-closed in X.

Let B be πgb-closed in X. Let B⊂ O where O is π-open in A. Then O = U∩A where U is π-open in X. This implies B⊂O = U∩A ⊂ U. Since B is πgb-closed in X, bcl(B) ⊂ U. Thus bcl_A(B)= A∩ bcl(B) ⊂ U∩A= O. Hence, B is πgb-closed relative to A.

Corollary 4.9: Let A be π-open, πgb-closed set. Then A∩F is πgb-closed whenever F∈ bC(X)

Proof: Since A is πgb- closed and π-open, then bcl(A)⊂A and thus A is b-closed. Hence A∩F is b-closed in X which implies A ∩F is πgb-closed in X.

Definition 4.10: A space (X, τ) is called a πgb-$T_{1/2}$ space if every πgb- closed set is b-closed.

Theorem 4.11:
(i) BO (τ) ⊂ πGBO(τ)
(ii) A space (X, τ) is πgb -$T_{1/2}$ iff BO(τ) = πGBO (τ).

Proof: (i) Let A be b-open, then X-A is b-closed so X-A is πgb-closed.Thus A is πgb-open.Hence BO (τ) ⊂ πGBO(τ)
(ii) **Necessity:** Let(X, τ) be πgb- $T_{1/2}$ space. Let A∈ πGBO(τ).Then X-A is πgb- closed. By hypothesis, X-A is b-closed thus A∈BO (τ). Thus πGBO (τ) = BO (τ).

Suffiency: Let BO(τ) = πGBO(τ).Let A be πgb-closed. Then X-A is πgb- open. X-A∈ πGBO(τ).X-A ∈ BO (τ).Hence A is b-closed. This implies (X, τ) is πgb- $T_{1/2}$ space.

Lemma 4.12: Let A be a subset of (X, τ) and x ∈ X. Then x ∈bcl (A) iff V∩Φ for every b-open set V containing x.

Theorem 4.13: For a topological space(X,) the following are equivalent
(i)X is πgb-$T_{1/2}$ space.
(ii)Every singleton set is either π-closed or b-open.

Proof: To prove (i) ⇒ (ii): Let X be a πgb-$T_{1/2}$ space .Let x ∈ X and assuming that {x} is not π- closed. Then clearly X-{x} is not π- open. Hence X-{x} is trivially a πgb- closed. Since X is πgb- $T_{1/2}$ space, X-{x} is b-closed. Therefore {x} is b-open.

(ii⇒(i): Assume every singleton of X is either π-closed or b-open. Let A be a πgb-closed set. Let {x} ∈ bcl(A).

Case (i): Let {x} be π- closed. Suppose {x} does not belong to A. Then {x}∈ bcl(A)-A. By theorem 3.8, {x} ∈A. Hence bcl(A) ⊂ A.

Case (ii): Let {x} be b-open .Since {x}∈ bcl(A),we have {x}∩A ≠ Φ implies {x} ∈ A. Therefore bcl (A) ⊂A. Therefore A is b-closed.

5. πgb- continuous and πgb- irresolute functions

Definition 5.1: A function f: (X, τ) → (Y, σ) is called πgb- continuous if every $f^{-1}(V)$ is πgb- closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.2: A function f: (X, τ) → (Y, σ) is called πgb- irresolute if $f^{-1}(V)$ is πgb- closed in(X, τ) for every gbr-closed set V in (Y, σ)

Proposition 5.3: Every πgb- irresolute function is πgb- continuous.

Remark 5.4: Converse of the above need not be true as seen in the following example.

Example 5.5: Consider X= {a, b, c}, τ ={ Φ,{a},{b},{a, b},X}.σ ={ Φ,{a},X}. Let f: (X, τ) → (X, σ) be the identity function. Then f is πgb-continuous but not πgb- irresolute.

Remark 5.6: Composition of two πgb-continuous functions need not be πgb-continuous

Example 5.7: Let X = {a, b, c, d}, τ = { Φ,{b},{c},{b, c},X}, σ ={ Φ,{a, b, d},X} η = { Φ,{a, d},X}. Define
f: (X, τ) → (X, σ) by f(a) = a, f(b) = c, f(c) = b ,f(d) =d. Define g: (X, σ) → (X, η) by g(a)= d, g(b) = c, g(c) = b, g(d) = a. Then f and g are πgb-continuous but g o f is not πgb-continuous.

Definition 5.8: A function f: X→Y is said to be pre b-closed if f (U) is b-closed in Y for each b-closed set in X.

Proposition 5.9: Let f: (X, τ) → (Y, σ) be π- irresolute and pre b-closed map. Then f(A) is πgb- closed in Y for every πgb- closed set A of X

Proof: Let A be πgb-closed in X. Let f(A) ⊂ V where V is π- open in Y. Then A⊂ $f^{-1}(V)$ and A is πgb-closed in X implies bcl(A) ⊂ $f^{-1}(V)$.Hence, f(bcl(A)) ⊂ V. Since f is pre b-closed, bcl(f(A)) ⊂ bcl(f(bcl(A))) = f(bcl(A))⊂V. Hence f (A) is πgb- closed in Y.

Definition 5.10: A topological space X is a πgb- space if every πgb- closed set is closed.

Proposition 5.11: Every πgb-space is πgb-$T_{1/2}$ space.

Theorem 5.12: Let f: (X, τ) → (Y, σ) be a function.
(1)If f is πgb- irresolute and X is πgb- $T_{1/2}$ space, then f is b-irresolute.
(2)If f is πgb- continuous and X is πgb- $T_{1/2}$ space, then f is b-continuous

Proof: (1) Let V be b-closed in Y. Since f is πgb-irresolute, $f^{-1}(V)$ is πgb-closed in X. Since X is πgb-$T_{1/2}$ space, $f^{-1}(V)$ is b-closed in X. Hence f is b-irresolute.

(2)Let V be closed in Y. Since f is πgb-continuous, $f^{-1}(V)$ is πgb- closed in X. By assumption, it is b-closed. Therefore f is b-continuous.

Definition 5.13[14]: A function f: (X, τ) → (Y, σ) is π-open map if f(F) is π-open map in Y for every π-open in X.

Theorem 5.14: If the bijective f :(X, τ) → (Y, σ) is b-irresolute and π-open map ,then f is πgb- irresolute.

Proof: Let V be πgb-closed in Y. Let $f^{-1}(V)$ ⊂ U where U is π- open in X. hen V⊂ f(U) and f(U) is π-open implies bcl(V) ⊂ f(U). Since f is b-irresolute, (f^{-1}(bcl(V))) is b-closed. Hence bcl($f^{-1}(V)$)⊂ bcl(f^{-1}(bcl(V))) = f^{-1}(bcl(V)) ⊂ U. Therefore, f is πgb- irresolute.

Theorem 5.15: If f: X→Y is π-open, b-irresolute, pre b-closed surjective function. If X is πgb- $T_{1/2}$ space, then Y is πgb-$T_{1/2}$ space.

Proof: Let F be a πgb-closed set in Y. Let $f^{-1}(F)$ ⊂U where U is π- open in X. Then F⊂ f(U) and F is a πgb-closed set in Y implies bcl(F)⊂f(U). Since f is b-irresolute, bcl($f^{-1}(F)$) ⊂ bcl(f^{-1}(bcl(F))) = f^{-1}(bcl(F))⊂U. Therefore $f^{-1}(F)$ is πgb-closed in X. Since X is πgb-$T_{1/2}$ space, $f^{-1}(F)$ is b-closed in X. Since f is pre b-closed, f($f^{-1}(F)$) = F is b-closed in Y. Hence Y is πgb-$T_{1/2}$ space.

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