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EFFECTS OF SURFACE TENSION ON FIXED-END NANO-BEAM SUBJECTED TO UNIFORM PRESSURE

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ABSTRACT

In this paper, the effects of surface tension on both ends fixed nano-beams subjected to uniform pressure are studied in the frame of surface elasticity theory. Ariy stress function methodology is used to obtain a set of analytical solutions. The analytical plane-stress solutions can be obtained for a uniformly pressure isotropic nano-beam with rectangular cross section classical boundary conditions. The results indicate some interesting characteristics, which are distinctly different from those in classical elasticity beam theory.

***Keywords**: Surface tension; Isotropic Nano-beam; Ariy stress function; Analytical solutions.*

1. INTRODUCTION

Micro- and nano-scale elements are more prone to the effects of surface tension than are their macro-scale counterparts. Due to the atoms near the free surfaces of a material experience a local environment distinct from those in the bulk. From the viewpoint of continuum mechanics, this difference can be described by such concepts as surface tension, surface energy, and surface constitutive relations. To account for the effect of surface tension in solid mechanics, Gurtin and Murdoch [1,2] adopted the classical continuum theory and formulated a surface elasticity model, where the surface of solids can be viewed as a 2D elastic membrane with different material constants adhering to the underlying bulk material without slipping. For some elementary deformation modes, the prediction of surface elasticity theory showed a good agreement with directly atomic simulation [3]. Therefore, the surface elasticity theory has been widely adopted to investigate the mechanical phenomena at nanoscale.

The plane stress problem of beams is very classical in elasticity theory and is encountered frequently in practical cases. Timoshenko and Goodier [4] investigated isotropic beams on uniform load and other cases of continuously loaded beams. Jiang and Ding [5] investigated orthotropic cantilever beams subjected to uniform load. For beams fixed at both ends subjected to uniform load, Gere and Timoshenko [6] presented the deflection and stress expressions with Euler-Bernoulli beam theory. Ahmed *et al*. [7] presented a numerical solution of fixed-end deep beams. Ding and Huang [8] obtained the analytical solution of both ends fixed nano-beams subjected to uniform load by Ariy stress function method. To the authors' knowledge, however, no literature for both ends fixed nano-beams with the effects of surface tension acted on uniform load has been published yet. Ariy stress function method was used to investigate fixed-end nano-beams with the effects of surface tension in plane stress subjected to uniform normal load, and obtain the stress and displacements analytical expressions.

In the present paper, the fundamental fixed-end nano-beam with surface tension effects subjected to pressures is considered. The Ariy stress function method is adopted to derive the analytical solutions of the both ends fixed nano-beams with surface tension effects. This method allows us to easily extend our analysis to problems involving fixed-end nano-beam subjected to uniformly distributed pressures on a finite region. The outline of the paper is organized as follows: The basic equations of surface elasticity theory and plan stress state are reviewed briefly in section 2 and 3. In section 4, Ariy stress function method is adopted to derive the fundamental solutions of a fixed-end nano-beam subjected to uniform pressure. The detailed results for the cases of uniformly distributed pressures are discussed in sections 5 and 6, and concluding remarks are presented in section 7.

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2. BASIC EQUATIONS OF SURFACE ELASTICITY

In surface elasticity theory, a surface is regarded as a negligibly thin membrane that has material constants different from the bulk material and is adhered to the bulk without slipping. The equilibrium and constitutive equations in the bulk of material are the same as those in classical elastic theory, but the presence of surface stresses gives rise to a non-classical boundary condition. For further details, the reader may refer to Gurtin *et al*. [1, 2]. Here only several basic equations of surface elasticity theory are reviewed.

In the absence of body force, the equilibrium equations, constitutive law and geometry relations in the bulk are as follows.

$$\sigma_{ij,j}=0 \tag{1}$$

$$\sigma_{ij}=2G\left(\varepsilon_{ij}+\frac{v}{1-2v}\varepsilon_{kk}\delta_{ij}\right) \tag{2}$$

where G and v are the shear modulus and Poisson's ratio of the bulk material, σ_{ij} and ε_{ij} are the stress tensor and strain tensor in the bulk material, respectively. Throughout the paper, Einstein's summation convention is adopted for all repeated Latin indices (1, 2, 3) and Greek indices (1, 2).

The strain tensor is related to the displacement vector u_i by

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \tag{3}$$

Assume that the surface of the material adheres perfectly to its bulk without slipping. Then the equilibrium conditions on the surface are expressed as

$$\sigma_{\beta\alpha}n_{\beta}+\sigma^{s}_{\beta\alpha,\beta}=0 \tag{4}$$

$$\sigma_{ij}n_in_j=\sigma^{s}_{\alpha\beta}\kappa_{\alpha\beta} \tag{5}$$

where n_i denotes the normal to the surface, $\kappa_{\alpha\beta}$ the curvature tensor of the surface, and $\sigma^{s}_{\alpha\beta}$ the surface stress tensor. The surface stress tensor $\sigma^{s}_{\alpha\beta}$ is related to the surface energy density $\gamma\left(\varepsilon_{\alpha\beta}\right)$ by

$$\sigma^{s}_{\alpha\beta}=\gamma\delta_{\alpha\beta}+\frac{\partial\gamma}{\partial\varepsilon_{\alpha\beta}} \tag{6}$$

The second term in Eq. (6) indicates the variation of the surface energy density with respect to the elastic strain. If the change of the atomic spacing during deformation is infinitesimal, the contribution from the second term to the surface stress will be negligibly small in comparison with the residual surface tension [9, 10].

Thus, the surface stresses are written as

$$\sigma^{s}_{\alpha\beta}=\gamma\delta_{\alpha\beta} \tag{7}$$

3. BASIC EQUATIONS IN PLAN STRESS STATE

As in classical theory of elasticity, we define the Airy stress function $\phi\left(x,z\right)$ by

$$\sigma_{11}=\frac{\partial^2\phi}{\partial z^2},\sigma_{33}=\frac{\partial^2\phi}{\partial x^2},\sigma_{13}=-\frac{\partial^2\phi}{\partial x\partial z} \tag{8}$$

For the considered plane problem, the equilibrium equations and Hooke**'s** law in the bulk reduce to

$$\frac{\partial\sigma_{11}}{\partial x}+\frac{\partial\sigma_{13}}{\partial z}=0,\frac{\partial\sigma_{13}}{\partial x}+\frac{\partial\sigma_{33}}{\partial z}=0 \tag{9}$$

$$\varepsilon_{11}=\frac{1}{E}\left[\sigma_{11}-\upsilon\sigma_{33}\right]$$

$$\varepsilon_{33}=\frac{1}{E}\left[\sigma_{33}-\upsilon\sigma_{11}\right]$$

$$\varepsilon_{13}=\frac{2\left(1+\upsilon\right)}{E}\sigma_{13} \tag{10}$$

Where E and υ are Young's modulus and Poisson ratio, respectively. The strains are related to the displacements by

$$\varepsilon_{11}=\frac{\partial u}{\partial x},\varepsilon_{33}=\frac{\partial w}{\partial z},\varepsilon_{13}=\frac{1}{2}(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}) \tag{11}$$

Which satisfy the following compatibility condition

$$\frac{\partial^2\varepsilon_{11}}{\partial z^2}+\frac{\partial^2\varepsilon_{33}}{\partial x^2}=2\frac{\partial^2\varepsilon_{13}}{\partial x\partial z} \tag{12}$$

Then the equilibrium equations in Eq. (12) are satisfied automatically and the compatibility equation in Eq. (8) becomes

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}\right)=0 \tag{13}$$

4. STRESS AND DISPLACEMENT

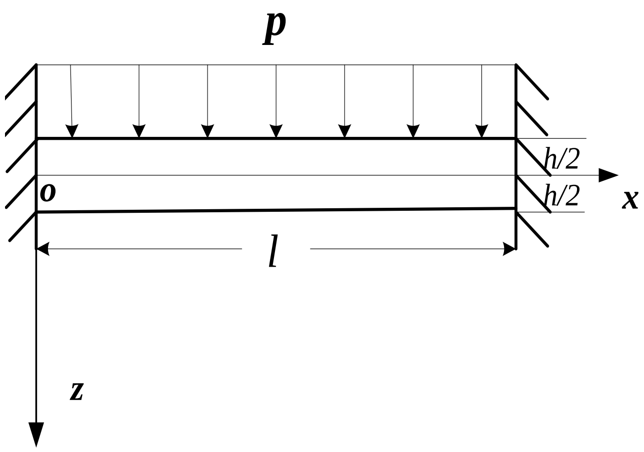

Figure-1: Fixed-end beam subjected to uniform normal load

Now we use the above surface elasticity theory to examine the influence of surface tension on a fixed-end nano-beam with unit width rectangular cross section subjected to uniform pressure $p(x)$. The length of the beam is l and height *h*. We refer to a Cartesian coordinate system (*o- xyz*), as shown in Fig. 1, where the *x* axis is along the surface, and the *z* axis perpendicular to the surface. Take the stress function in the following form of a bi-harmonic polynomial with 7 terms

$$\phi(x,z)=a\left(\frac{1}{5}z^5-x^2z^3\right)+bxz^3+cz^3+dz^2+ex^2z+fxz+gx^2 \tag{14}$$

where *a, b, c, d, e, f* and *g* are unknown constants to be determined. The substitution of Eq.(14) into Eq.(8) gives

$$\sigma_{11}=2a\left(2z^3-3x^2z\right)+6bxz+6cz+2d \tag{15}$$

$$\sigma_{33}=-2az^3+2ez+2g \tag{16}$$

$$\sigma_{13}=6axz^2-3bz^2-2ex-f \tag{17}$$

Substituting Eqs. (15) and (16) into the first two equations in Eq. (10), and then integrating them with respect to *x* and *z*, respectively, yields the displacement components $u(x,z)$ and $w(x,z)$ as

$$u(x,z)=\frac{1}{E}\left[2a(2+\upsilon)xz^3-\left(2ax^3-3bx^2-6cx+2\upsilon ex\right)z+2dx-2\upsilon gx\right]+A(z) \tag{18}$$

$$w(x,z)=\frac{1}{E}\left[-\frac{a}{2}(1+2\upsilon)z^4+\left(3a\upsilon x^2-3b\upsilon x-3\upsilon c+e\right)z^2-2d\upsilon z+2gz\right]+B(x) \tag{19}$$

where $A(z)$ and $B(x)$ are functions of *x* and *z*, respectively.

Substituting Eqs. (18), (19) and (17) into the third equation in Eq. (10), we have

$$\frac{1}{E}\left[3b(2+\upsilon)z^2+2(1+\upsilon)f\right]+A'(z)+\frac{1}{E}\left[-2ax^3+3bx^2+6cx+2(2+\upsilon)ex\right]+B'(x)=0 \quad (20)$$

Eqs. (11) and (12) can be obtained from Eq. (10)

$$\begin{aligned}&\frac{1}{E}\left[3b(2+\upsilon)z^2+2(1+\upsilon)f\right]+A'(z)=\omega\\&\frac{1}{E}\left[-2ax^3+3bx^2+6cx+2(2+\upsilon)ex\right]+B'(x)=-\omega\end{aligned} \quad (21)$$

where ω is an arbitrary constant. It is noted that, $A(z)$ and $B(x)$ can be obtained by integrating Eq.(21). Substituting $A(z)$ and $B(x)$ into Eqs. (18) and (19), we have

$$\begin{aligned}u(x,z)=&\frac{1}{E}\left[2a(2+\upsilon)xz^3-\left(2ax^3-3bx^2-6cx+2\upsilon ex\right)z+2dx-2\upsilon gx\right]\\&-\frac{1}{E}\left[b(2+\upsilon)z^3+2(1+\upsilon)fz\right]+\omega z+u_0\end{aligned} \quad (22)$$

$$\begin{aligned}w(x,z)=&\frac{1}{E}\left[-2a(1+2\upsilon)z^4+\left(3a\upsilon x^2-3b\upsilon x-3\upsilon c+e\right)z^2-2d\upsilon z+2gz\right]\\&+\frac{1}{E}\left[\frac{1}{2}ax^4-bx^3-3cx^2-(2+\upsilon)ex^2\right]-\omega x+w_0\end{aligned} \quad (23)$$

where arbitrary constants u_0 and w_0 denote the translation and rotation of rigid body, respectively.

5. FIXED-END NANO-BEAMS SUBJECTED TO UNIFORM PRESSURE

In the case, there is uniform pressure p_0 acting over the region $|x|\le l$ (Fig.1). The plane-strain conditions are assumed with $\varepsilon_{2i}=0$, and the contact is assumed to be frictionless. In this case, the stress boundary conditions Eqs. (4) and (5) on the contact surface ($z=-h/2$) are simplified to

$$\sigma_{13}=0 \quad (24)$$

$$p(x)+\sigma_{33}=-\frac{\gamma}{R(x)} \quad (25)$$

where $p(x)$ is the external pressure applied on the surface, and $R(x)$ is the curvature radius of the deformed surface which is given by

$$\frac{1}{R(x)}=\frac{\partial^2 w(x,0)}{\partial x^2}=\frac{\partial^2 B(x)}{\partial x^2}=\frac{1}{E}\left[6ax^2-6bx-6c-2(2+\upsilon)e\right] \quad (26)$$

Timoshenko and Goodier [4] presented the methods for dealing with the boundary conditions for fixed-end beams. The method is to treat the displacement boundary conditions as, (1) $z=h/2,\sigma_{33}=0$, (2) $z=-h/2,\sigma_{33}=-p-\gamma/R(x)$, (3) $z=\pm h/2,\sigma_{13}=0$, (4) $x=0,z=0$ point and $x=l,z=0$ point, $u=0,w=0$, $\partial u/\partial z=0;\partial w/\partial x=0$. By substituting Eqs. (22) and (21) of the displacement components and the Eqs. (24), (25) and (26) of stress components into corresponding boundary conditions, 10 algebraic equations can be obtained and all the unknown constants can be determined as

$$a=\frac{4p\exp(1)\left[12+(5v+13)v\right]}{4l^2\left[28+\left(37+13v\right)v\right]\gamma+13h^3\exp(1)v+(5v^2+12h^2)\left[1+\exp(1)\right]h\exp(1)}$$

$$b=\frac{4p\exp(1)\left[8+(3v+7)v\right]}{4l^2\left[28+\left(37+13v\right)v\right]\gamma+13h^3\exp(1)v+(5v^2+12h^2)\left[1+\exp(1)\right]h\exp(1)}$$

$$c=\frac{2p\exp(1)l^2v}{4l^2\left[28+\left(37+13v\right)v\right]\gamma+13h^3\exp(1)v+(5v^2+12h^2)\left[1+\exp(1)\right]h\exp(1)}$$

$$d=\frac{p\exp(1)hv\left[(5v^2+13v+12)h^2-2(2+v)l^2\right]}{8l^2\left[28+\left(37+13v\right)v\right]\gamma+26h^3\exp(1)v+2(5v^2+12h^2)\left[1+\exp(1)\right]h\exp(1)} \tag{27}$$

$$e=-\frac{2p\exp(1)l^2(2+v)}{4l^2\left[28+\left(37+13v\right)v\right]\gamma+13h^3\exp(1)v+(5v^2+12h^2)\left[1+\exp(1)\right]h\exp(1)}$$

$$f=\frac{p\exp(1)l\left\{3h^2\left[16+(19+7v)v\right]+4(2+v)l^2\right\}}{4l^2\left[28+\left(37+13v\right)v\right]\gamma+13h^3\exp(1)v+(5v^2+12h^2)\left[1+\exp(1)\right]h\exp(1)}$$

$$g=\frac{p\exp(1)h\left\{\left[12+(13+5v)v\right]\exp(1)h^2-2(2+v)l^2\right\}}{8l^2\left[28+\left(37+13v\right)v\right]\gamma+26h^3\exp(1)v+2(5v^2+12h^2)\left[1+\exp(1)\right]h\exp(1)}$$

Substituting Eq. (27) into Eq. (15), (16), (17), (21) and (22), the stress and displacement components are then obtained

$$\begin{aligned}\sigma_{11}=&(p\exp(1)(5\exp(1)h^3v^2+13\exp(1)h^3v-40v^2z^3+12\exp(1)h^3-2hl^2v-72l^2vxz\\&-120v^2x^2z+80v^2z^3-4hl^2v+12l^2zv+168lvxz-312vx^2z+208vz^3+192lxz\\&-288x^2z+192z^3))\big/((52l^2v^2+148l^2v+112l^2)\gamma\\&+(5h^3v^2+13h^3v+12h^3)(\exp(1)+\exp(2)))\end{aligned} \tag{28}$$

$$\begin{aligned}\sigma_{13}=&-(p\exp(1)(21h^2lv^2+36lv^2z^2-120v^2xz^2+57h^2lv+4l^3v-4l^2vx+84lvz^2\\&-312vxz^2+48h^2l+8l^2x+96lz^2-288xz^2))\big/((52l^2v^2+148l^2v+112l^2)\gamma\\&+(5h^3v^2+13h^3v+12h^3)(\exp(1)+\exp(2)))\end{aligned} \tag{29}$$

$$\begin{aligned}\sigma_{33}=&(p\exp(1)(5\exp(1)h^3v^2+13\exp(1)h^3v-40v^2z^3+12\exp(1)h^3-2hl^2v\\&-4l^2vz-104vz^3-4hl^2-8l^2z-96z^3))\big/((52l^2v^2+148l^2v+112l^2)\gamma\\&+(5h^3v^2+13h^3v+12h^3)(\exp(1)+\exp(2)))\end{aligned} \tag{30}$$

$$\begin{aligned}u=&-(2p\exp(1)z(21h^2lv^3+6lv^3z^2-20v^3xz^2+78h^2lv^2+4l^3v^2-2l^2v^2x-18l^2x^2\\&+26lv^2z^2+20v^2x^3-92v^2xz^2+105h^2lv+12l^2v-10l^2vx-42lvx^2+44lvz^2\\&+52vx^3-152vxz^2+48h^2l+8l^3-48lx^2+32lz^2+48x^3\\&-96xz^2))\big/E((52l^2v^2+148l^2v+112l^2)\gamma\\&+(5h^3v^2+13h^3v+12h^3)(\exp(1)+\exp(2)))\end{aligned} \tag{31}$$

$$w = -(p\exp(1)(-2l^2vx^2 + 80v^3z^4 - 10v^2x^4 + 248v^2z^4 - 26vx^4 + 296vz^4 + 4l^2z^2 - 2hl^2v^3z + 36lv^3xz^2 - 4hl^2z + 84lv^2xz^2 + 2hl^2vz + 96lvxz^2 + 5\exp(1)h^3v^4z + 13\exp(1)h^3v^3z + 7\exp(1)h^3v^2z - 13\exp(1)h^3vz - 12\exp(1)h^3z + 32lx^3 - 8lx^2 - 40v^3x^2z^2 - 2l^2v^2x^2 + 6l^2v^2z + 12lv^2x^3 - 104v^2x^2z^2 + 2l^2vz^2 + 28lvx^3 - 96vx^2z^2 + 4hl^2z - 24x^4 + 96z^4))/E((52l^2v^2 + 148l^2v + 112l^2)\gamma + (5h^3v^2 + 13h^3v + 12h^3)(\exp(1) + \exp(2))) \quad (32)$$

6. NUMERICAL RESULTS

It is instructive to examine the effects of the surface tension on the stresses or displacement and compare them with the classical elasticity theory. In what follows, we set $E = 2\times10^{11}, v = 0.3, z/h = 0.04, l/h = 2.$ [11].

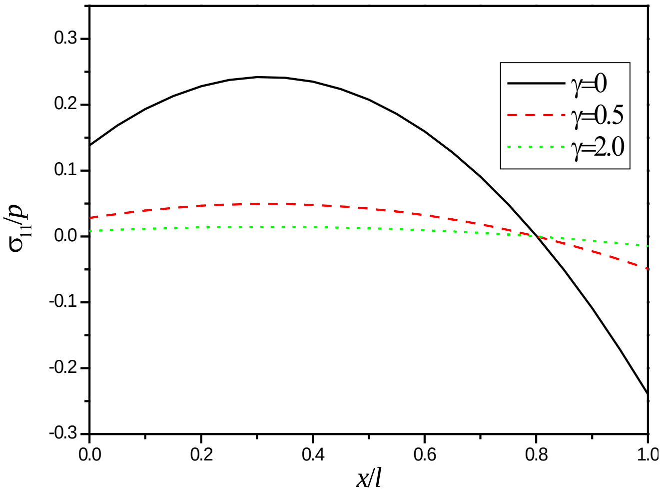

Figure-2: Distribution of the stress σ_{11} under uniform loading.

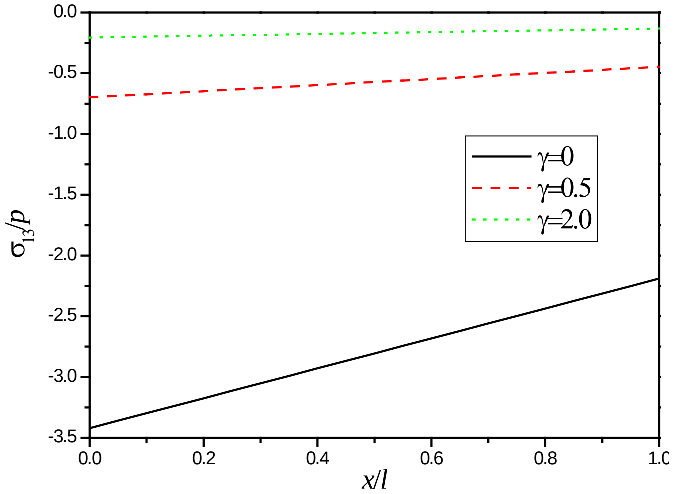

Figure-3: Distribution of the stress σ_{13} under uniform loading.

For the illustration of the effects of surface tension, γ is taken as 0, 0.5, and 2 in our calculations. Figures 2-4 show the distribution of the stresses σ_{11}, σ_{13} and σ_{33}, where the solution of $\gamma=0$ is consistent with the classical elastic result. When the parameter γ, that is, of the order of nanometers, the influence of surface tension is evidently significant. It can be seen from Figure 2 that the horizontal stress transits continuously across the loading zone. In addition, the stress curve of the degree of curvature gradually slow down with the increase of γ. It is also found in Figure 3 that shear stress changes smoothly across the loading boundary zone, and shear stress is approximate linear relation about *x*. Figure 4 show the normal stress σ_{33} is a fixed value on the loading region, as is decreases with the increase of surface tension.

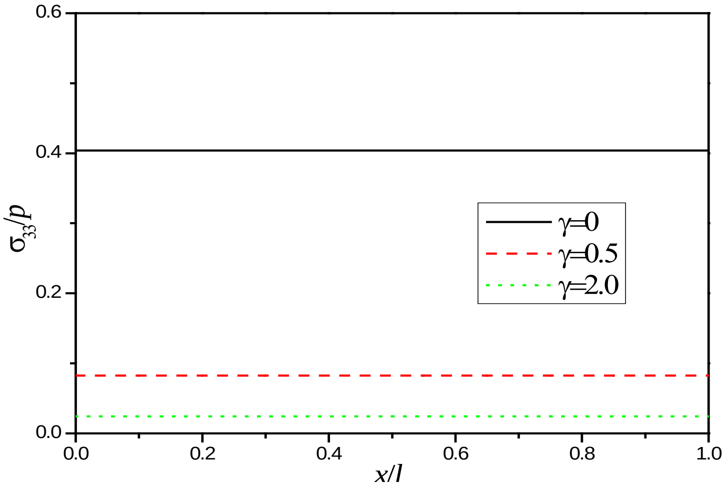

Figure-4: Distribution of the stress σ_{33} under uniform loading.

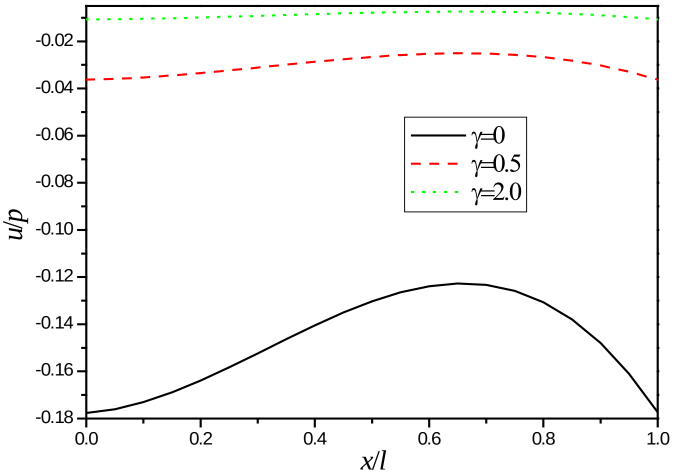

Figure-5: Distribution of the displacement u under uniform loading.

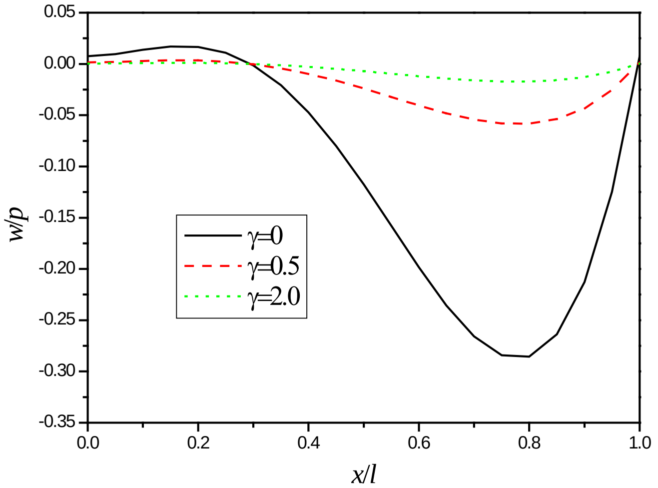

Figure-6: Distribution of the displacement w under uniform loading.

For various values of surface tension, the horizontal displacement u on the loading region is plotted in Figure 5. It is seen that the slope of the deformed is continuous everywhere, as is decreases and more gentle with the increase of surface tension. Figure 6 show the distribution of the normal displacement, which also reveal that the normal displacement is continuous everywhere on the loading zone. In addition, It is found out that the normal displacement is decrease with the surface tension.

7. CONCLUSIONS

In the present paper, we consider the influence of surface tension on the mechanical behavior of the fixed-end nano-beam subjected to uniform pressure. Through the Ariy stress function methodology, the general analytical solutions of nano-beam are derived rigorously. It is found out that surface elasticity theory illuminates some interesting characteristics of fixed-end beam on nanoscales, which are distinctly different form the classical solutions of elasticity without surface tension. The stress and the deformation displacement show a significant dependence on the surface tension. Our results display that for nano-beam problems, the classical elasticity theory predicts some unreasonable results and therefore the effects of surface tension should be accounted.

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