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ON ALMOST BOOLEAN ALGEBRAS AND RINGS

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ABSTRACT

In this paper we make a detailed study on the concepts of Almost Boolean algebras and Almost Boolean rings. Mainly, we prove that the class of Almost Boolean algebras and the class of Almost Boolean rings form categories and we establish an equivalence between them.

Key words: *Almost Distributive lattice (ADL), Almost Boolean algebra (ABA), Almost Boolean ring (ABR) and categorical equivalence.*

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1. INTRODUCTION

The concept of an Almost Boolean Algebra (ABA) was introduced by U.M. Swamy and G.C Rao [6], while studying a common abstraction of several lattice theoretic generalizations of Boolean algebra and other extensions of Boolean lattice Theory (N.V Subramanyamn [2], [3] and [4]). Also Swamy and Rao [6] have introduced a ring theoretic generalization of Boolean ring and called it an Almost Boolean ring (ABR).

It is well known that there is a duality between Boolean algebras and Boolean rings. In this paper we discuss in detail, a categorical equivalence between ABA's and ABR's. In section 3, we give equivalent conditions for an Almost distributive lattice A with a maximal element to be an Almost Boolean algebra and define a binary operation $*$ on an ABA$(A, \wedge, \vee, 0)$ and discuss properties of it interrelation to $\wedge$ and $\vee$. In section 4, we make a detailed study on ABR's. In section 5, we extend the notions of morphism and congruence on ABA's and ABR's and prove that these coincide whether we view it as an ABA as well as an ABR. In section 6, we observe that the class of ABA's is equationally definable and hence a variety (that is, it is closed under the formation of products, subalgebras and homomorphic images); on the other hand, it is a category in which the objects are ABA's and the morphisms are the homomorphisms. Similarly, the class of ABR's is also a variety as well as a category. Finally, we establish an equivalence between the categories of ABA's and ABR's.

2. PRELIMINARIES

Definition: 2.1 An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (ADL) if it satisfies the following conditions for all a, b and c in A.

(1) $0 \wedge a = 0$
(2) $a \vee 0 = a$
(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(4) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(6) $(a \vee b) \wedge b = b$

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An ADL A is called an associative ADL if the operation $\vee$ is associative. Throughout this paper, by an ADL we always mean an associative ADL only. Most of the preliminaries presented in this section are taken from [6].

Example: 2.2 Any distributive lattice bounded below is an ADL.

Example: 2.3 Let X be a nonempty set and fix an arbitrarily chosen element $0 \in X$. For any a and $b \in X$, define

$$a \wedge b = \begin{cases} 0 & \text{if } a = 0 \\ b & \text{if } a \neq 0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} b & \text{if } a = 0 \\ a & \text{if } a \neq 0 \end{cases}$$

Then (X, ∧, ∨, 0) is an ADL and is called a discrete ADL.

Definition: 2.4 Let $A = (A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in$ A, define

$$a \leq b \quad \text{if} \quad a = a \wedge b \ (\Leftrightarrow a \vee b = b).$$

Then $\leq$ is a partial order on A with respect to which 0 is the smallest element in A.

Theorem: 2.5 The following hold good for any elements a, b and c in an ADL A.

(1) $a \wedge 0 = 0 = 0 \wedge a$ and $a \vee 0 = a = 0 \vee a$
(2) $a \wedge a = a = a \vee a$
(3) $a \wedge b \leq b \leq b \vee a$
(4) $a \wedge b = a \Leftrightarrow a \vee b = b$ and $a \wedge b = b \Leftrightarrow a \vee b = a$
(5) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (i.e., $\wedge$ is associative)
(6) $a \vee (b \vee a) = a \vee b$
(7) $a \leq b \Rightarrow a \wedge b = a = b \wedge a$ and $a \vee b = b = b \vee a$
(8) $a \wedge b \wedge c = b \wedge a \wedge c$
(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(10) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ whenever $a \wedge b = 0$
(11) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$
(12) $a \wedge b = \inf\{a, b\} \Leftrightarrow a \vee b = \text{Sup}\{a, b\}$

Theorem: 2.6 Let (A, ∧, ∨, 0) be an ADL and $\leq$ be the induced partial order. The following are equivalent to each other for any element $m \in A$.

(1) m is a maximal element in $(A, \leq)$
(2) $m \wedge a = a$ for all $a \in A$
(3) $m \vee a = m$ for all $a \in A$
(4) $a \vee m$ is maximal for all $a \in A$.

3. ALMOST BOOLEAN ALGEBRA

Any bounded complemented distributive lattice $(L, \wedge, \vee, ', 0, 1)$ is known as a Boolean algebra. In this section, we discuss a special class of ADL's. First, we begin with the following.

Theorem: 3.1 Let A be an ADL. Then the following are equivalent to each other.
(1) For any $a, b \in A$, there exists $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$.
(2) For any $a \leq b$ in A, $[a, b]$ is a complemented lattice.
(3) For any $a \in A$, $[0, a]$ is complemented lattice.

Proof:
(1) ⟹ (2): Let $a \leq b$ in A and let $y \in [a, b]$. It is known that $[a, b]$ is a lattice. By (1), there exists $x \in A$ such that $y \wedge x = 0$ and $y \vee x = y \vee b = b$.

Now $y \wedge (a \vee x) = (y \wedge a) \vee (y \wedge x) = a \vee 0 = a$ and $y \vee (a \vee x) = (y \vee a) \vee x = y \vee x = b$.

Also, $a \leq a \vee x \leq b$ (by 2.5 (10), $x \vee y = y \vee x = b$ and hence $x \leq b$).

Therefore, $a \vee x$ is a complement of y in $[a, b]$. Thus $[a, b]$ is a complemented lattice.

(2) ⟹ (3): is clear.

(3) ⟹ (1): follows from the fact that $a \leq a \vee b$ and hence $a \in [0, a \vee b]$ and by (2) a will have a complement in $[0, a \vee b]$.

Definition: 3.2 A non trivial ADL A is called an Almost Boolean algebra (ABA) if it has a maximal element and satisfies one, and hence all of the equivalent conditions given in the above theorem.

Examples: 3.3

(1) Any nontrivial discrete ADL (as given in 2.3) is an Almost Boolean algebra in which each nonzero element is a maximal element. Note that any discrete ADL with more than two elements is not a Boolean algebra, since $x \wedge y = y$ and $y \wedge x = x$ for any nonzero elements x and y.

(2) Let $(R, +, \cdot, 0, 1)$ be a commutative regular ring with unity 1. For any a and $b \in$ R, define

$$a \wedge b = a_0 b \quad \text{and} \quad a \vee b = a + b - a_0 b,$$

where a_0 is the unique idempotent in R such that $aR = a_0 R$.

Then $(R, \wedge, \vee, 0)$ is an ABA in which the maximal elements in $(R, \leq)$ are precisely the units.

The following examples show that the existence of maximal elements and the relative complements are independent from each other.

(1) Let X be an infinite set and A be the set of all finite subsets of X. Then A is a distributive lattice with empty set as zero element under the usual set theoretic operations (and hence an ADL) satisfying the conditions in the Theorem 3.1. However, A has no maximal element.

(2) Let $\mathbb{N}$ be the set of all non-negative integers. Define $\wedge$ and $\vee$ as follows:

$$a \wedge b = gcd\{a, b\} \quad \text{and} \quad a \vee b = lcm\,\{a, b\}.$$

Then $(\mathbb{N}, \wedge, \vee, 1)$ is a bounded distributive lattice in which 1 is the smallest element and 0 is the largest element. However, this does not satisfy the conditions given in Theorem 3.1.

Theorem: 3.4 Let A be an ADL with a maximal element. Then the following are equivalent to each other.

(1) A is an Almost Boolean algebra
(2) For any $a \in A$, there exists $b \in A$ such that $a \wedge b = 0$ and $a \vee b$ is maximal
(3) $[0, m]$ is a Boolean algebra for all maximal elements m
(4) There exist a maximal element m such that $[0, m]$ is a Boolean algebra

Proof:

(1) ⟹ (2): Let $a \in A$ and m be a maximal element in A. Since A is an ABA, and by the theorem 3.1 (1), there exists $b \in A$ such that $a \wedge b = 0$ and $a \vee b = a \vee m$ and hence $a \vee b$ is maximal.

(2) ⟹ (3): Assume (2). Let m be a maximal element in A and $a \in [0, m]$. By (2), there exists $b \in A$ such that $a \wedge b = 0$ and $a \vee b$ is maximal. Let $x = b \wedge m$. Then $x \in [0, m]$ and

$$a \wedge x = a \wedge b \wedge m = 0 \text{ and } a \vee x = a \vee (b \wedge m) = (a \vee b) \wedge (a \vee m) = a \vee m = m$$

(Since $a \vee b$ is maximal and $a \leq m$).

Therefore, x is the complement of a in $[0, m]$. Thus $[0, m]$ is a Boolean algebra.

(3) ⟹ (4): is clear. To prove (4)⟹ (1), let m be a maximal element in A such that $[0, m]$ is a Boolean algebra.

Let $a \in$ A and $x \in [0, a]$. Then $x \wedge m \leq m$ and hence there exits $y \leq m$ such that $(x \wedge m) \wedge y = 0$ and $(x \wedge m) \vee y = m$.

Now, $x \wedge (y \wedge a) = (x \wedge y) \wedge a = (x \wedge (m \wedge y)) \wedge a = 0 \wedge a = 0$ since m is maximal

$$\begin{aligned}
\text{and} \quad x \vee (y \wedge a) &= (x \wedge a) \vee (y \wedge a) \quad (\text{since } x \leq a)\\
&= (x \vee y) \wedge a = (y \vee x) \wedge a \text{ (by 2.5 (9))}\\
&= (y \vee x) \wedge (y \vee m) \wedge a \quad (\text{since y} \vee m \text{ is maximal}\\
&= (y \vee (x \wedge m)) \wedge a = ((x \wedge m) \vee y) \wedge a \quad ((\text{by 2.5 (9)})\\
&= m \wedge a = a \text{ (since m is maximal).}
\end{aligned}$$

Therefore $y \wedge a$ is the complement of x in $[0, a]$. Thus $[0, a]$ is a complemented lattice for all $a \in A$ and hence A is an Almost Boolean algebra.

The following is a more general one, it establishes that the Boolean algebra $[0, m]$ is independent (up to isomorphism) of the maximal element m in Almost Boolean algebra A.

Theorem: 3.5 Let A be an ADL and m and n be maximal elements in A. Then the lattices $[0, m]$ and $[0, n]$ are isomorphic to each other. Moreover, the Boolean algebras $[0, m]$ and $[0, n]$ are isomorphic when Aan Almost Boolean algebra is.

Theorem: 3.6 Let $(A, \wedge, \vee, 0)$ be an ABA. Then for any a and b in A, there exists unique $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$.

Proof: Let a and $b \in A$. By theorem 3.1(1), there exists $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$. It can be easily verified that x is the complement of a in the Boolean algebra $[0, a \vee b]$ and hence x is unique.

Let us define a binary operation * on an ABA A and observe certain properties of this, which will be useful in proving that A has a structure almost like a Boolean ring.

Definition: 3.7 The unique element x in the above theorem will be denoted by $a * b$. That is, for any elements a and b in an Almost Boolean algebra A, $a * b$ is the unique element in A satisfying the equations

$$a \wedge (a * b) = 0 \text{ and } a \vee (a * b) = a \vee b.$$

Theorem: 3.8 Let $(A, \wedge, \vee, 0)$ be an ABA. The following hold good for any elements a and b in A.

(1) $0 * a = a$ and $a * 0 = 0$
(2) $a * a = 0$
(3) $(a * b) \wedge a = 0$
(4) $(a * b) \vee a = a \vee b$
(5) $a * b$ is the complement of a in $[0, a \vee b]$
(6) $(a * b) * a = a$
(7) $a * (a * b) = a * b$
(8) $(a * b) * b = a \wedge b$
(9) $b * (a * b) = 0$
(10) $a * b = 0 \Leftrightarrow a = a \vee b$

Proof:
(1) Follows from $0 \wedge a = 0$ and $0 \vee a = a = a \vee 0$.
(2) We have $a \wedge 0 = 0$ and $a \vee 0 = a = a \vee a$.
(3) Since $a \wedge (a * b) = 0$, $(a * b) \wedge a = 0$ (by 2.5 (10))
(4) By (3) and 2.5 (10), $(a * b) \vee a = a \vee (a * b) = a \vee b$
(5) This is observed in the proof of theorem 3.1(1)
(6) and (7) are consequences of (3) above.
(8) We have $(a * b) \wedge (a \wedge b) = a \wedge (a * b) \wedge b = 0 \wedge b = 0$

and $(a * b) \vee (a \wedge b) = ((a * b) \vee a) \wedge ((a * b) \vee b) = (a \vee (a * b)) \wedge ((a * b) \vee b)$
$= (a \vee b) \wedge ((a * b) \vee b) = [(a \vee b) \wedge (a * b)] \vee [(a \vee b) \wedge b]$
$= (a * b) \vee b$ (since $a * b \leq a \vee b$)

and therefore, $(a * b) * b = a \wedge b$.

(9) We have $b \wedge 0 = 0$ and $b \vee 0 = b \vee (a \wedge (a * b)) = ((b \vee a) \wedge (b \vee (a * b)) = ((b \vee a) \wedge b) \vee ((b \vee a) \wedge (a * b))$
$= b \vee ((a \vee b) \wedge (a * b)) = b \vee (a * b)$ (since $a * b \leq a \vee b$)

and hence $b * (a * b) = 0$.

(10) $a * b = 0 \Longrightarrow a \vee 0 = a \vee b$ (by (4) above) $\Longrightarrow a = a \vee b$.
Also $a = a \vee b \Longrightarrow a \wedge 0 = 0$ and $a \vee 0 = a \vee b \Longrightarrow 0 = a * b$.

The following theorem gives certain inter relationships between binary operations $\wedge$, $\vee$and $*$.

Theorem: 3.9 The following hold for any elements a, b and c in an Almost Boolean algebra $A = (A, \wedge, \vee, 0)$.

(1) $a * (b \vee c) = (a * b) \vee (a * c)$
(2) $a * (b \wedge c) = (a * b) \wedge (a * c)$
(3) $(a \vee b) * c = (a * c) \wedge (b * c)$
(4) $(a \wedge b) * c = (a * c) \vee (b * c)$

Proof: (1) Put $x = (a * b) \vee (a * c)$. We have

$a \wedge x = a \wedge ((a * b) \vee (a * c)) = (a \wedge (a * b)) \vee (a \wedge (a * c)) = 0 \vee 0 = 0$

and $a \vee x = a \vee (a * b) \vee (a * c) = a \vee b \vee (a * c) = a \vee b \vee a \vee (a * c)$ (by 2.5 (6))
$= a \vee b \vee a \vee c = a \vee b \vee c = a \vee (b \vee c)$.

and therefore $a * (b \vee c) = x = (a * b) \vee (a * c)$.

(2) Put $y = (a * b) \wedge (a * c)$ then $a \wedge y = a \wedge (a * b) \wedge (a * c) = 0 \wedge (a * c) = 0$

and $a \vee y = a \vee ((a * b) \wedge (a * c)) = (a \vee (a * b)) \wedge (a \vee (a * c)) = (a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$

and hence $a * (b \wedge c) = y = (a * b) \wedge (a * c)$

(3) Put $z = (a * c) \wedge (b * c)$. We have $(a \vee b) \wedge z = (a \vee b) \wedge (a * c) \wedge (b * c)$
$= (a \wedge (a * c) \wedge (b * c)) \vee (b \wedge (a * c) \wedge (b * c))$
$= (0 \wedge (b * c)) \vee ((a * c) \wedge b \wedge (b * c)) = 0 \vee 0 = 0$

and $a \vee b \vee z = a \vee b \vee ((a * c) \wedge (b * c)) = (a \vee b \vee (a * c)) \wedge (a \vee b \vee (b * c))$
$= (b \vee a \vee (a * c)) \wedge (a \vee (b \vee c)) = (b \vee (a \vee c)) \wedge ((a \vee b) \vee c)$
$= ((a \vee b) \vee c) \wedge ((a \vee b) \vee c) = (a \vee b) \vee c$.

Thus, $(a \vee b) * c = (a * c) \wedge (b * c)$.
(4) can be proved by using the similar technique as above.

Corollary: 3.10 The following hold for any elements a and b in an Almost Boolean algebra A.
(1) $(a \wedge b) * a = b * a$
(2) $(a \wedge b) * b = a * b$
(3) $(a \vee b) * a = 0$
(4) $(a \vee b) * b = 0$

Theorem: 3.11 Let $A = (A, \wedge, \vee, 0)$ be an Almost Boolean algebra and $a, b, c \in A$. Then the following hold.
(1) $(a * b) \wedge c = a * (b \wedge c)$
(2) $(a * b) \wedge b = a * b$ and $(a * b) \vee b = b$
(3) $a \wedge (b * c) = b * (a \wedge c)$
(4) $(a * b) \vee c = (a * (b \vee c)) \vee c$
(5) $a \vee (b * c) = (a * b) * (a \vee c)$

4. ALMOST BOOLEAN RINGS

U.M. Swamy and G.C. Rao [6] have introduced a ring theoretic generalization of Boolean ring and termed it as an Almost Boolean ring (ABR) and they have exhibited a one-to-one correspondence between ABA's and ABR's. In this section, we make a detailed study of this correspondence which leads to establishing a categorical equivalence between ABA's and ABR's. Now, we introduce the notion of an Almost Boolean ring.

Definition: 4.1 An algebra $(A, +, \cdot, 0)$ is called an Almost Boolean ring (ABR) if (A is a nonempty set, + and . are binary operations on A and 0 is a distinguished element in A) it satisfies the following, for any a, b and c in A.
(1) $a + 0 = a = 0 + a$
(2) $a + a = 0$
(3) $a \cdot a\ a$
(4) $(a \cdot b) \cdot c\ \ a \cdot (b \cdot c)$
(5) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
(6) $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$
(7) $((a + b) + c) \cdot d = (a + (b + c)) \cdot d$
(8) There is *a* left identity for $\cdot$ in A

The proof of the following theorem follows from the theorem 3.8.

Theorem: 4.2 Let $(A, \wedge, \vee, 0)$ be an ABA. Define binary operations + and $\cdot$ on A by
$$a + b = (a * b) \vee (b * a) \text{ and } a \cdot b = a \wedge b.$$
Then $(A, +, \cdot, 0)$ is an ABR.

Corollary: 4.3 Let X be a set with at least two elementsand 0 and m be two distinct elements of X. For any x and $y \in X$, define
$$x + y = \begin{cases} y & if \ \ x = 0 \\ x & if \ \ y = 0 \\ 0 & if \ \ x \neq 0 \text{ and } y \neq 0 \end{cases} \quad \text{and} \quad x \cdot y = \begin{cases} 0 & if \ \ x = 0 \\ y & if \ \ x \neq 0 \end{cases}$$

Then $(X, +, \cdot, 0, m)$ is an ABR.

Corollary: 4.4 Le t $(R, +, \cdot, 0, 1)$ be a commutative regular ring with unity. For any x and $y \in R$, define
$$x \oplus y = x + y - x_0 y - y_0 x \text{ and } x \odot y = x_0 \cdot y,$$
where x_0 is the unique idempotent such that $xR = x_0 R$. Then $(R, \oplus, \odot, 0, 1)$ is an ABR.

Theorem: 4.5 Let $(R, +, \cdot, 0, 1)$ be an ABR. Then the following hold good for any a, b and $c \in$ R.
(1) $0 \cdot a = 0 = a \cdot 0$
(2) Ra = $\{x \cdot a \colon x \in \mathrm{R}\}$ is a Boolean ring under the operations induced by + and $\cdot$.
(3) $(a \cdot b) \cdot c = (b \cdot a) \cdot c$
(4) $(a + b) \cdot c = (b + a) \cdot c$
(5) $a \cdot b = 0 \Leftrightarrow b \cdot a = 0$.
(6) $a + b = b + a$.

The following is a converse of theorem 4.2.

Theorem: 4.6 Let (R, +, $\cdot$, 0) be an ABR. For any x and $y \in$ R, define
$$x \wedge y = x \cdot y \text{ and } x \vee y = x + (y + xy).$$
Then $(R, \wedge, \vee, 0)$ is an ABA.

Theorem: 4.7 Let $(B, \wedge, \vee, 0)$ be an ABA and $(B, +, \cdot, 0)$ be the corresponding ABR. Then the following are equivalent to each other.

1. (B, ∧, ∨, 0) is a Boolean algebra.
2. ∧ is commutative.
3. ∨ is commutative.
4. (B, ≤) has a unique maximal element.
5. \+ is associative.
6. (B, +, ·, 0) is a Boolean ring.

5. MORPHISMS AND CONGRUENCES

In [1], Stone proved that a morphism between two Boolean algebras is also a morphism between the corresponding Boolean rings and vice-versa and that the same is true for congruences. Here, we extend these concepts to ABA's and ABR's and we prove that these coincide in an ABA as well as an ABR.

Now, formally we introduce the notions of morphisms and congruences on ABA's and ABR's.

Definition: 5.1 Let (A, ∧, ∨, 0) and (B, ∧, ∨, 0) be ABA's. A mapping *f*: A → B is called a **homomorphism** (or simply, a morphism) if $f(a \wedge b) = f(a) \wedge f(b)$, $f(a \vee b) = f(a) \vee f(b)$ and $f(0) = 0$, for any a and b∈A.

Observe that $f(0) = 0$ if and only if $f(a * b) = f(a) * f(b)$, for all $a, b \in A$. If f is an epimorphism, then $f(m)$ is a maximal element in B for any maximal element m in A.

Note that, without being surjective, a morphism may not carry maximal elements onto maximal elements, even in the case of Boolean algebras.

Definition: 5.2 Let $(A, \wedge, \vee, 0)$ be an ABA. An equivalence relation θ on A is said to be a congruence if

$$(a,b) \text{ and } (c,d) \in \theta \Longrightarrow (a \wedge c, b \wedge d) \in \theta \text{ and } (a \vee c, b \vee d) \in \theta.$$

It can be observed that for any congruence θ on an ABA A, the Quotient $A/_\theta$ is also an ABA under the induced operations ∧ and ∨ on $A/_\theta$ defined by:

$$\theta(a) \wedge \theta(b) = \theta(a \wedge b) \text{ and } \theta(a) \vee \theta(b) = \theta(a \vee b),$$

Where, $\theta(a) = \{b \in A : (a,b) \in \theta\}$, in which $\theta(0)$ is the zero element and $\theta(m)$ is maximal for any maximal m in A. Also, the class of congruence relation on A form a complete lattice under the inclusion ordering.

The following gives an inter-connection between homomorphisms and congruences on ABA's.

Theorem: 5.3 Let A be an ABA and θ a binary relation on A. Then θ is a congruence on A if and only if θ is the kernel of a homomorphism of A into some ABAB.

Theorem: 5.4 Let A and B be ABA's and θ a congruence on A, and $f: A \to B$ an epimorphism such that $\ker f = \theta$, where, $ker\ f = \{(a,b) \in A \times A : f(a) = f(b)\}$. Then there is an isomorphism $\alpha : A/_\theta \to B$ such that $\alpha \circ \beta = f$, where $\beta : A \to A/_\theta$ is the natural map.

Similarly, the above theorem holds good in the case of ABR's.

6. THE CATEGORICAL EQUIVALENCE

In this section, we prove that the classes of ABA's and ABR's form categories and establish an equivalence between them. Let us recall from theorems 4.2 and 4.6 that an ABA can be made as an ABR and vice-versa. In the following we give that these two processes are inverses to each other.

Theorem: 6.1 Let $(A, \wedge, \vee, 0)$ be an ABA and $R(A) = (A, +, \cdot, 0)$ be the corresponding ABR.
Also, Let $A(R(A)) = (A, \underline{\wedge}, \underline{\vee}, 0)$ be the ABA obtained from R(A). Then $A = A(R(A))$.

Theorem: 6.2 Let $(R, +, \cdot, 0)$ be an ABR and $A(R) = (R, \wedge, \vee, 0)$ be the corresponding ABA. Also,
let $R(A(R)) = (R, \oplus, \odot, 0)$ be the ABR obtained from $A(R)$. Then $R = R(A(R))$.

Corollary: 6.3 The correspondences $A \longmapsto R(A)$ and $R \longmapsto A(R)$ are inverses to each other and establish a one-to-one correspondence between the class of ABA's and the class of ABR's.

Theorem: 6.4 Let A_1 and A_2 be ABA's and $R(A_1)$ and $R(A_2)$ be the corresponding ABR's. Also, let $f\colon A_1 \to A_2$ be a mapping. Then f is a homomorphism of ABA's if and only if it is a homomorphism of ABR's.

Proof: Suppose that $f\colon A_1 \to A_2$ is a homomorphism of ABA's.

Now for any $a,b \in A_1$, $f(a+b) = f((a*b) \vee (b*a)) = f(a*b) \vee f(b*a)$
$$= \big(f(a) * f(b)\big) \vee \big(f(b) * f(a)\big)$$
$$= f(a) + f(b)$$
and $f(a \cdot b) = f(a \wedge b) = f(a) \wedge f(b) = f(a) \cdot f(b)$.

Therefore $f\colon R(A_1) \to R(A_2)$ is a homomorphism of ABR's.

Conversely suppose that $f\colon R(A_1) \to R(A_2)$ is a homomorphism of ABR's. Then for any a and b in A_1,
$$f(0) = f(a+a) = f(a) + f(a) = 0,$$
$$f(a \wedge b) = f(a \cdot b) = f(a) \cdot f(b) = f(a) \wedge f(b)$$
$$\text{and } f(a \vee b) = f\big(a + (b + a \cdot b)\big) = f(a) + f(b + a \cdot b)$$
$$= f(a) + \big(f(b) + f(a) \cdot f(b)\big) = f(a) \vee f(b).$$
Thus $f\colon A_1 \to A_2$ is a homomorphism of ABA's.

Corollary: 6.5 $A \mapsto R(A)$ and $R \mapsto A(R)$ establish an equivalence between the category of ABA's and that of ABR's.

Theorem: 6.6 Let $(A, \wedge, \vee, 0)$ be an Almost Boolean algebra and $R(A) = (A, +, \cdot, 0)$ be the corresponding Almost Boolean ring. Let θ be an equivalence relation on A. Then is θ is a congruence on A if and only if it is a congruence on $R(A)$.

Proof: Assume that θ is compatible with A. Since the binary operations $\wedge$ and $\cdot$ are the same, we have to only prove that θ is compatible with $\vee$ if and only if it is so with +. Suppose that θ is compatible with ∨. Then, the quotient $A/_\theta$ is an ABA in which $\theta(a) * \theta(b) = \theta(a*b)$ for any $a, b \in A$.

This implies that θ is compatible with the operation $*$ also.

Now, for any $a, b, c, d \in A$, $(a,b), (c,d) \in \theta \Rightarrow (a*c, b*d), (c*a, d*b) \in \theta$
$$\Rightarrow ((a*c) \vee (c*a), (b*d) \vee (d*b)) \in \theta$$
$$\Rightarrow (a+c, b+d) \in \theta.$$
Therefore θ is compatible with +.

Conversely supposethat θ is compatible with +.

Then $(a,b),\ (c,d) \in \theta \Rightarrow (a,b),\ (c,d),\ (ac, bd) \in \theta$
$$\Rightarrow \big(a + (c + ac), b + (d + bd)\big) \in \theta$$
$$\Rightarrow (a \vee c, b \vee d) \in \theta.$$
Therefore, θ is compatible with ∨. Thus θ is a congruence on the Almost Boolean algebra A if and only if it is so on the Almost Boolean ring $R(A)$.

REFERENCES

1. Stone M.H.: *Theory of representations for Boolean algebras*, Trans. Amer.Math. Soc., 40(1936), 37-111.
2. Subrahmanyam N.V.: *Structure theory for generalized Boolean rings*, Math. Ann., 141(1960), 297-310.
3. Subrahmanyam N.V.: *An extension of Boolean lattice theory*, Math. Ann., 151(1963), 332-345.
4. Subrahmanyam N.V.: *Boolean vector spaces I*, Math.Z., 83(1964), 422-433.
5. Swamy U.M.: *Representation of Universal algebras by Sheaves*, Proc. Amer. Math. Soc., 45(1974), 53-58.
6. SwamyU.M., Rao G.C.: *Almost distributive lattices*, J. Australian Math. Soc., Series- A, 31(1981), 77- 91.
7. Swamy U.M., Rao G.C., G.N. Rao G.N.: *Pseudo- complementation on Almost distributive lattices*, Southeast Asian Bull. Math. 24(2000)95-104.
8. Swamy U.M., RaoG.C., Rao G.N.: *Stone Almost distributive lattices*, Southeast Asian Bull. Math. 27(2003) 513-526.

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