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DOMINATION PARAMETERS OF SOME GRAPHS AND ITS REALIZATION

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ABSTRACT

The domination parameters of a graph G of order n has been already introduced in [9]. It is defined as $D \subseteq V(G)$ is a dominating set of G, if every vertex $v \in V\text{-}D$ is adjacent to atleast one vertex in D. In this paper we established various domination parameters of some graphs such as path, cycle, wheel, star, r-corona and complete bipartite graph with m, n vertices. Also established the relation between this parameters and illustrated an example for some graphs which is deviated from its general formula.

1. INTRODUCTION

A graph G = (V, E), were V is a finite set of elements called vertices and E is a set of unordered pairs of distinct vertices of G called edges. The degree of a vertex v in G is the number of edges incident on it. A graph G is said to be k-regular if all its vertices are of degree k. Every pair of its vertices are adjacent in G, is said to be complete, the complete graph on 'n' vertices is denoted by K_n.

A graph G is said to be bipartite or bigraph if the vertex set of V(G) can be partitioned in to two subsets X and Y such that every edge of G has one in X and the other end in Y. A bipartite graph G with | X | = m and | Y| = n is said to be complete if every element in one partition is adjacent with all elements of the other partition and is denoted by $K_{m,n}$. The graph $K_{1,n}$ is called a Star graph.

Let u, and v be the vertices of a graph G, a u-v walk of G is an alternating sequences $u = u_0, e_1, u_1, u_2, \dots, u_{n-1}\, e_n\, u_n = v$ of vertices and edges beginning with vertex u and ending with vertex v such that $e_i = u_{i-1}u_i$, for all i = 1, 2, …, n. The number of edges in a walk is called its length. A walk in which all the vertices are distinct is called a path. A path on 'n' vertices is denoted by P_n. A closed path is called a cycle, a cycle on 'n' vertices are denoted by C_n. Let G = (V, E) be a simple connected graph, for any vertex $v \in V$, the open neighborhood is the set $N(v) = \{u \in V / uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subset V$, the open neighborhood of S is $N(s) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$.

Definition 1.1: A set $D \subseteq V$ is a dominating set of G if every vertex $v \in V - D$ is adjacent to atleast one vertex of D. We call a dominating set D is a minimal if there is no dominating set $D' \subseteq V(G)$ with $D' \subset D$ and $D' \neq D$. Further we call a dominating set D is minimum if these is no dominating set $D' \subseteq V(G)$ with $|D'| < |D|$. The cardinality of a minimum dominating set is called the domination number denoted by $\gamma(G)$ and the minimum dominating set D of G is also called a γ- set.

Definition 1.2: A dominating set D is said to be a total dominating set if every vertex in V is adjacent to some vertex in D. The total domination number of G. denoted by γ_t (G) is the minimum cardinality of a total dominating set.

Definition 1.3: A dominating set D of a graph G is an independent dominating set, if the induced subgraph <D> has no edges. The independent domination number γ_i (G) is the minimum cardinality of a independent dominating set.

Definition 1.4: A dominating Set D is said to be connected dominating set, if the induced subgraph <D> is connected. The connected domination number γ_c(G) is the minimum cardinality of a connected dominating set.

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Definition 1.5: A dominating Set D of a graph G is said to be a paired dominating set if the induced subgraph <D> contains atleast one perfect matching, paired domination number γ_p (G) is the minimum cardinality of a paired dominating set.

Definition 1.6: A dominating Set D of G is a split dominating set if the induced sub graph <V – D> disconnected Split domination number γ_s (G) is the minimum cardinality of a split dominating set.

Definition 1.7: A dominating Set D of G is a non split dominating set, if the induced sub graph <V – D> is connected. Non split domination number γ_{ns} (G) is the minimum cardinality of a non split dominating set.

Definition 1.8: Let D be a γ- set of G. A dominating set D^1 contained in V – D is called an inverse dominating set of G with respect to D. The inverse domination number γ' (G) is the minimum cardinality of all inverse dominating set of G, the vertices of γ' (G) is called γ'- set.

Definition 1.9: A dominating set D of a graph G is called a global dominating set, if D is also a dominating set of $\overline{G}$. The global domination number γ_g (G) in the minimum cardinality of a global dominating set.

Definition 1.10: A dominating set D is called a perfect dominating set, if every vertex in V – D in adjacent to exactly one vertex in D. The perfect domination number γ_{pr}(G) is the minimum cardinality of a perfect dominating set.

Definition 1.11: If D = {x} is a dominating set of G, then x is called a dominating vertex of G. A vertex v∈ V(G) is said to be a γ - required vertex of G, if v lies in every γ- set of G.

Definition 1.12: Let x be any real value, then its upper sealing of x is denoted as φ x κ and is defined

$$\varphi x \kappa = \begin{cases} x & \text{if } x \text{ is an integer} \\ \text{k}, & \text{where k is an integer lies in the interval } x < \text{k} < x + 1 \end{cases}$$

the lower sealing of x is denoted as λxμ and is defined by

$$\lambda x \mu = \begin{cases} x & \text{if } x \text{ is an integer} \\ \text{k}, & \text{where k is an integer lies in the interval } x - 1 < \text{k} < x \end{cases}$$

Lemma 2.1: Let G be a connected graph with δ(G) ≥ 2, them γ(G) + γ'(G) = n if and only if G = P_4 or C_4.

Lemma 2.2: Let G be a connected graph with δ = 1 and Δ = n then γ (G) + γ' (G) = n + 1 if and only if G = $k_{1,\,n}$.

Lemma 2.3: For any tree with n ≥ 2 with more then two pendent vertices then there exists a vertex v ∈V such that γ (T – v) = γ (T).

Lemma 2.4: For any path P_n, γ (p_n) ≤ γ' (p_n) ∀ n ≥ 3.

Proof: Since P_n is a path with n vertices then

$$\gamma\ (P_n) = \begin{cases} \gamma'(P_n) - 1 & \text{if } n = 3k\ \ \forall\ \ k = 1, 2 \ldots \\ \gamma'(P_n) & \text{otherwise} \end{cases}$$

therefore, $\gamma (P_n) \le \gamma' (P_n)$. ∀ n ≥ 2

Note: Let G be a path of length n then

$$\left.\begin{array}{l} \gamma(P_n) = \left\lceil \dfrac{n}{3} \right\rceil \\ \gamma'(P_n) = \left\lfloor \dfrac{n}{3} \right\rfloor + 1 \end{array}\right\} \forall\ \ n > 3$$

Lemma 2.5: Let G be a cycle of length four then $\gamma(G) = \gamma'(G) = \gamma_t(G) = \gamma_c(G) = \gamma_p(G) = \gamma_s(G) = \gamma_g(G) = \gamma_{ns}(G) = \gamma_i(G) = 2$.

Proof: Let v_1, v_2, v_3 and v_4 are the vertices of C_4, each vertex v_i connected with v_{i+1}, i = 1, 2, 3 and v_4 is connected with v_1 in G. Hence v_1, v_3 and v_2v_4 are the edges in $\overline{G}$. Let D={v_1, v_2} be the vertices of G. Clearly D satisfies the conditions for total domination, connected dominating, paired domination, global domination and non split domination therefore, $\gamma(G) = \gamma_t(G) = \gamma_c(G) = \gamma_p(G) = \gamma_g(G) = \gamma_{ns}(G) = 2$.

Let D′= {v_3, v_4} satisfies the condition for the inverse domination, therefore, $\gamma'(G) = 2$. Let D_1 = {v_1, v_3} satisfies the condition for independent domination and split domination, therefore, $\gamma_i(G) = \gamma_s(G) = 2$.

Hence,

$$\gamma(G) = \gamma_t(G) = \gamma_p(G) = \gamma_g(G) = \gamma_{ns}(G) = \gamma'(G) = \gamma_s(G) = \gamma_i(G) = 2$$

Note: C_4 is the smallest simple connected graph which satisfies the conditions of all dominations parameters with its cardinality is two.

Lemma 2.6: For any complete graph K_n.

$$\gamma(G) = \gamma'(G) = \gamma_i(G) = \gamma_{ns}(G) = 1 \text{ and } \gamma_t(G) = \gamma_p(G) = 2$$

Theorem 2.7: For any path P_n, n > 2 the domination parameters satisfies the following.

i) $\gamma(G) = \gamma_i(G) = \gamma_s(G) = \gamma_g(G) = \left\lceil \frac{n}{3} \right\rceil$

ii) $\gamma'(G) = \left\lfloor \frac{n}{3} \right\rfloor + 1$

iii) $\gamma_t(G) = \begin{cases} 2k+1 & \text{if } n = 4k + 1 \;\; k = 1, 2, \ldots \\ 2\left\lceil \frac{n}{4} \right\rceil & \text{otherwise} \end{cases}$

iv) $\gamma_p(G) = 2\left\lceil \frac{n}{4} \right\rceil$

v) $\gamma_c(G) = \gamma_{ns}(G) = n - 2$

Proof:

i) Let P_n be the path and its vertices are denoted by $v_1, v_2, v_3, \ldots, v_n$, for all n > 2. Now subdivide the path into sub graphs $G_1, G_2, G_2, \ldots, G_k$ such that each sub graphs G_i, i = 1, 2, . . . ,k containing three consecutive vertices from the beginning.

That is G_1 = {v_1, v_2, v_3} ; G_2 = {v_4, v_5, v_6}; G_3 = {v_7, v_8, v_9}, . . . , G_k = {v_{n-2}, v_{n-1}, v_n} if n = 3k. collect all the vertices {$v_{2k+i} \in G_k$, k = 1, 2, 3, . . . , and i = k – 1 is the required minimum dominating set of G.

That is, D ={$v_2, v_5, v_8, v_{11}, \ldots, v_{n-1}$}, we have collected exactly one element form each G_i, i = 1, 2, . . ., k.

Hence, | D | = k

$$= \frac{n}{3} \quad [\text{Since } n = 3k]$$

$$= \left\lceil \frac{n}{3} \right\rceil \quad [\frac{n}{3} \text{ is an integer } \left\lceil \frac{n}{3} \right\rceil = \frac{n}{3}]$$

Suppose n = 3k – 1, then the last partition G_k contains only last two vertices. That is, G_k = {v_{n-1}, v_n} then S = {v_{2k+i}/ k = 1, 2 . . . , i = k – 1 and 2k + i ≤ n – 2} now S = {$v_2, v_5, v_8, \ldots, v_{n-3}$} then D = S ∪ {$v_{n-1}$} and D = S ∪ {$v_n$} is the required minimum dominating sets of G with cardinality k.

$$n = (k-1)\,3 + 2 \Rightarrow \frac{n}{3} = (k-1) + \frac{2}{3}$$

$$\Rightarrow \left\lceil \frac{n}{3} \right\rceil = \lceil k-1 \rceil + \left\lceil \frac{2}{3} \right\rceil = (k-1) + 1 \Rightarrow |D| = \left\lceil \frac{n}{3} \right\rceil.$$

If n=3k–2, as in the above case all sub graphs G_i, i = 1, 2, . . , $\overline{k-1}$. Containing three vertices and the last partition G_k containing the only vertex v_n.

Now $S = \{v_{2k+i} / k = 1, 2, 3, \ldots; 2k + i \le n-1 \text{ and } i = k-1\}$ then $D = S \cup G_k$ is the minimum dominating set with cardinality k (i)

We have $n = 3k - 2 \Rightarrow n = 3(k-1) + 1$

$$\left\lceil \frac{n}{3} \right\rceil = k - 1 + \frac{1}{3}$$

$$\left\lceil \frac{n}{3} \right\rceil = k - 1 + \left\lceil \frac{1}{3} \right\rceil \Rightarrow \left\lceil \frac{n}{3} \right\rceil = |D|$$

Therefore, in each case $\gamma(G) = \left\lceil \frac{n}{3} \right\rceil$

In all cases the induced subgraph D are independent in G. therefore, $\gamma_i(G) = \gamma(G) = \left\lceil \frac{n}{3} \right\rceil$

Since, P_n is a tree the induced subgraph $<V - D>$ is disconnected $\Rightarrow \gamma_i(G) = \gamma(G) = \left\lceil \frac{n}{3} \right\rceil$

Clearly D is dominating set of $\overline{G}$.

Hence, $\gamma(G) = \gamma_i(G) = \gamma_s(G) = \gamma_g(G) = \left\lceil \frac{n}{3} \right\rceil$

(ii) **Case (i):** if n = 3k by case (i) $D = \{v_{2k+i} / k = 1, 2, 3, \ldots; i = k-1 \text{ and } 2k + i \le n\}$

That is, $D = \{v_2, v_5, v_8, \ldots, v_{n-1}\}$ is the minimum dominating set of P_n.

Now choose the elements of $v \in V - D$ such that

$S = \{v_{3i+i} / i = 0, 1, 2, \ldots \text{ and } 3i + 1 \le n\}$ then

$S' = \{v_1, v_4, v_7, v_{10}, \ldots, v_{n-2}\}$, now v_n is not adjacent to any vertex $v_i \in S'$

Let $D' = S' \cup \{v_n\}$, $D' = \{v_1, v_4, v_7, \ldots, v_{n-2}, v_n\}$, now we have selected one vertex from each subgraph G_k, k = 1, 2, 3, . k – 1 such that $v_{3i+1} \in G_{i+1}$ and two elements v_{n-2} and v_n from G_k.

Therefore,
$$\begin{aligned} |D'| &= k - 1 + 2 \\ &= k + 1 \\ &= \frac{n}{3} + 1 \end{aligned}$$

$$|D'| = \left\lfloor \frac{n}{3} \right\rfloor + 1 \quad \left[\because n = 3k; \ \frac{n}{3} = \left\lfloor \frac{n}{3} \right\rfloor \right]$$

Case (ii): n=3k–2 by case (i) the sub graphs G_i, i = 1, . . ., k – 1 containing exactly three vertices and G_k contains only one vertex $\{v_n\}$ and $D = \{v_2, v_5, \ldots, v_{n-2}, v_n\}$ by case (i) $D' = \{v_{3i+i} / i = 0, 1, 2, \ldots \text{ and } 3i + 1 < n\}$

$D' = \{v_1, v_4, v_7, \ldots, v_{n-3}, v_{n-1}\}$ is the required inverse dominating set of G.

$|D'| = (k-2)\,1 + 2 = k$

We have,
$$\begin{aligned} n &= 3k - 2 \\ &= 3(k-1) + 1 \end{aligned}$$

$$\frac{n}{3} = k - 1 + \frac{1}{3}$$

$$\left\lfloor \frac{n}{3} \right\rfloor = k - 1 + \left\lfloor \frac{1}{3} \right\rfloor$$

$$\left\lfloor \frac{n}{3} \right\rfloor = k - 1 \qquad \left\{ \because \left\lfloor \frac{1}{3} \right\rfloor = 0 \right\}$$

$$\Rightarrow k = \left\lfloor \frac{n}{3} \right\rfloor + 1$$

$$\Rightarrow |D| = \left\lfloor \frac{n}{3} \right\rfloor + 1$$

Case (iii): $n = 3k - 1$ by the previous argument each subgraph G_i, $i = 1, 2, \ldots, k - 1$ containing exactly three vertices and G_k contains two vertices $\{v_{n-1}, v_n\}$ then

$D = \{v_{2k+i} / k = 1, 2, \ldots;\ 2k + i \le n \text{ and } i = k - 1\}$ is the dominating set of G and

$D' = \{v_{3i+1} / i = 0, 1, 2, \ldots \text{ and } 3i + 1 \le n\}$

$D' = \{v_1, v_4, v_7, \ldots, v_{n-1}\}$ is the inverse dominating set with respect to D in G and $|D'| = k$. that is,

$n = 3(k-1) + 2$

$$\frac{n}{3} = k - 1 + \frac{2}{3} \Rightarrow \left\lfloor \frac{n}{3} \right\rfloor = k - 1 + 0$$

$$\Rightarrow k = \left\lfloor \frac{n}{3} \right\rfloor + 1$$

$$\Rightarrow |D'| = \left\lfloor \frac{n}{3} \right\rfloor + 1$$

Hence, $\gamma'(P_n) = \left\lfloor \frac{n}{3} \right\rfloor + 1$ for all n.

(iii) if $n = 4k + 1$. Divide the vertices of G into k partition such that each partition G_i, $i = 1, \ldots, k - 1$ containing four vertices and the last partition G_k contains exactly five vertices

then $G_1 = \{v_1, v_2, v_3, v_4\}$: $G_2 = \{v_5, v_4, v_7, v_8\}; \ldots$

$G_k = \{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$

Choose middle two vertices from G_i, $i = 1 \ldots k$ and three vertices from G_k.

$D_t(G) = \{v_{2k+i}, v_{3k+1} / k = 1, 2, 3 \ldots;\ i = k - 1 \text{ and } 3k + i \le n\} \cup \{v_{n-1}\}$ is the required minimum total dominating set of G and $|D_t(G)| = 2(k-1) + 3 \Rightarrow |D_t(G)| = 2k + 1$

That is, $\gamma_t(P_n) = 2k + 1$ if $n = 4k + 1$

if $n \ne 4k+1$, then the vertices is of the form $G_1 = \{v_1, v_2, v_3, v_4\}$: $G_2 = \{v_5, v_6, v_7, v_8\}; \ldots$ the last partition G_k is either $\{v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$ or $G_k = \{v_{n-6}, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$

In both G_k we have to select two pair of vertices for the total domination of G_k and a pair of vertices $\{(G_{2k+i}, v_{3k+i}) / k = 1, 2, \ldots;\ i = k - 1 \text{ and } 3k + i \le n - 6\}$.

Therefore, $|D_t(G)| = (k-1)2 + 4$

$= 2k + 2$ (ii)

If $n = 4k + 2 \Rightarrow \frac{n}{4} = k + \frac{2}{4}$

$$\left\lceil \frac{n}{4} \right\rceil = k + \left\lceil \frac{2}{4} \right\rceil$$

$$= k + 1$$

$$\Rightarrow \quad 2(k+1) = 2\left\lceil \frac{n}{4} \right\rceil$$

$$\Rightarrow \quad |D_t(G)| = 2\left\lceil \frac{n}{4} \right\rceil \quad [\,\square\, |D_t(G)| = k + 1]$$

if $n = 4k + 3$

$$\frac{n}{4} = k + \frac{3}{4} \Rightarrow \left\lceil \frac{n}{4} \right\rceil = k + \left\lceil \frac{3}{4} \right\rceil$$

$$= k + 1$$

$$\Rightarrow 2(k+1) = 2\left\lceil \frac{n}{4} \right\rceil$$

$$\Rightarrow |D_t(G)| = 2\left\lceil \frac{n}{4} \right\rceil$$

$$\Rightarrow \quad \gamma_t(Pn) = \begin{cases} 2k+1 & \text{if } n = 4k + 1 \\ 2\left\lceil \frac{n}{4} \right\rceil & \text{otherwise} \end{cases}$$

(iv) by (iii) divide the vertices of Pn in to k subsets such that each subset containing four vertices is of the form $G_1 = \{v_1, v_2, v_3, v_4\}$; $G_2 = \{v_5, v_6, v_7, v_8\}$ . . . then the k^{th} partition G_k is any one of the following.

$$G_k = \begin{cases} v_{n-3}, v_{n-2}, v_{n-1}, v_n & \text{if } n = 4k \\ v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n & \text{if } n = 4k+1 \\ v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n & \text{if } n = 4k+2 \\ v_{n-6}, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n & \text{if } n = 4k+3 \end{cases}$$

If n = 4k, the pair of middle two vertices in each partition of G_i, i = 1 . . . k is the required minimum paired dominating set of G.

That is $D_p(G) = \{v_{2i}, v_{2i+1} / i = 1, 3, 5, \ldots, \text{ and } 2i+1 \le n\}$ then $D_p(G) = \{v_2, v_3, v_6, v_7, v_{10}, v_{11}, \ldots, v_{n-2}, v_{n-1}\}$

$$\Rightarrow |D_p(G)| = 2k$$

$$= 2.\left(\frac{n}{4}\right) \qquad [\because n = 4k]$$

$$= 2\left\lceil \frac{n}{4} \right\rceil \qquad \left[\because \frac{n}{4} = \left\lceil \frac{n}{4} \right\rceil\right]$$

If n = 4k + 1, select middle two vertices from each partition G_i, i = 1 . . . , $\overline{k-1}$, and any two pair of vertices which forms a paired dominating set of G_k.

$$|Dp(G)| = 2(k-1) + 4$$

$$= 2k + 2$$

$$= 2(k+1)$$

$$= 2\left(\frac{n-1}{4} + 1\right) \qquad [\because n = 4k+1]$$

$$= 2\left(\frac{n+3}{4}\right)$$

$$= 2\left\lceil \frac{n}{4} \right\rceil \qquad \left[\because n-1 \text{ is a multiple of } 4 \quad \frac{n+3}{4} = \left\lceil \frac{n}{4} \right\rceil\right]$$

$$\Rightarrow \gamma_p(G) = 2\left\lceil \frac{n}{4} \right\rceil$$

In similar, if n = 4k + 2

$$|D_p(G)| = 2(k-1) + 4$$

$$= 2(k+1)$$

$$= 2\left(\frac{n-2}{4} + 1\right) \qquad [\because n = 4k+2]$$

$$= 2\left(\frac{n+2}{4}\right) \qquad \left[\because n-2 \text{ is a multiple of } 4 \quad \frac{n+2}{4} = \left\lceil \frac{n}{4} \right\rceil\right]$$

$$|D_p(G)| = 2\left\lceil \frac{n}{4} \right\rceil$$

In similar $|D_p(G)| = 2\left\lceil \frac{n}{4} \right\rceil$, if n = 4k + 3.

Hence, $\gamma_p(G) = 2\left\lceil \frac{n}{4} \right\rceil \ \forall\ n \ge 2$

(v) Let $G = \{v_1, v_2, \ldots v_n\}$ be the vertex of P_n then by definition

(vi) $D_c(G) = G - \{v_1, v_n\}$ and $D_{ns}(G) = G - \{v_1, v_n\}$

Therefore, $\gamma_c(G) = \gamma_{ns}(G) = n - 2$ for all n > 3

Result 2.8: If G is a path with n vertices them

$$\gamma \le \gamma_i \le \gamma_s \le \gamma_g \le \gamma' \le \gamma_+ \le \gamma_{ns} \le \gamma_c$$

The following table represents the values of the various domination parameters of P_n, $n \le 10$.

	γ	γ_i	γ_s	γ_g	γ'	γ_t	γ_p	γ_{ns}	γ_c
P_3	1	1	1	2	2	2	2	2	2
P_4	2	2	2	2	2	2	2	2	2
P_5	2	2	2	2	2	3	4	3	3
P_6	2	2	2	2	3	4	4	4	4
P_7	3	3	3	3	3	4	4	5	5
P_8	3	3	3	3	3	4	4	6	6
P_9	3	3	3	3	4	5	6	7	7
P_{10}	4	4	4	4	4	6	6	8	8

Corollary 2.9: For any integer $n \ge 4$, the only graph which satisfy the condition

$$\gamma (P_n) = \gamma_i(P_n) = \gamma_s(P_n) = \gamma_g(P_n) = \gamma' (P_n) = \gamma_t (P_n) = \gamma_p(P_n) = \gamma_{ns} (P_n) = \gamma_c(P_n) = 2 \text{ is } P_4$$

Proposition 2.10: For any integer $n \ge 4$

$$\gamma(P_n) = \gamma_i(P_n) = \gamma_s(P_n) = \gamma'(P_n) = \gamma_g (p_n) = 2 \text{ iff } n = 4, 5$$

Proof: by (i) of Theorem 2.7

$$\gamma (P_n) = \gamma_i(P_n) = \gamma_s(P_n) = \gamma_g (P_n) = \left\lceil \frac{n}{3} \right\rceil$$

$n = 4, 5,$

$$\left\lceil \frac{4}{3} \right\rceil = 2; \left\lceil \frac{5}{3} \right\rceil = 2$$

$\Rightarrow$ $\gamma (P_n) = \gamma_i(P_n) = \gamma_s(P_n) = \gamma_g (P_n) = 2$ by (ii) of 2.7 $n = 4,5$

$$\gamma^1(P_4) = \left\lfloor \frac{4}{3} \right\rfloor + 1 = 2; \ \gamma^1(P_5) = \left\lfloor \frac{5}{3} \right\rfloor + 1 = 2$$

Hence, $\gamma (P_n) = \gamma_i(P_n) = \gamma_s(P_n) = \gamma' (P_n) = \gamma_g(P_n) = 2$ iff $n = 4, 5$.

Theorem 2.11: For any path P_n, $n > 3$, $G = \overline{P_n}$ then

$$\gamma (G) = \gamma_i(G) = \gamma' (G) = \gamma_t (G) = \gamma_{ns}(G) = \gamma_c (G) = 2$$

Proof: Let $v_1, v_2, v_3, \ldots v_n$ be the vertices of the graph P_n and each vertex v_i, $i = 2, \ldots, n-1$ is connected with v_{i-1} and v_i, v_1 and v_n are connected only with v_2 and v_{n-1} respectively, that is

$d(v_1) = d(v_n) = 1$ and $d(v_i) = 2$ for all $i = 2, 3, 4, \ldots, \overline{n-1}$.

Let $G = \overline{P_n}$; then $d(v_i) = d(v_n) = n - 2$, $v_1, v_n \in G$

$d(v_i) = n - 3 \ \forall v_i \in G; i = 2, \ldots, n - 1$.

In G the vertices v_1 and v_4 are connected with all vertices of G other than v_2 and v_{n-1} respectively.

Now $\{v_1, v_n\}$ and any vertex set $\{v_i, v_j\}_{i \ne j}$ is the minimum dominating set of $G = \overline{P_n}$.

Since $v_i v_{i+1} \in E(P_n)$

which are independent in G and is the minimum independent dominating set of G.

Since $n > 3$ for any set $[v_i, v_j\}$ is a dominating set of P_n then any pair of vertices $\{v_l, v_m\} \in V - D$ is the inverse dominating set of G. Since $v_1 v_n \notin E(\overline{G_n})$, therefore,$\{v_1, v_n\}$ is the total, connected and non split dominating set of G.

Hence, $\gamma (G) = \gamma_i(G) = \gamma'(G) = \gamma_+ (G) = \gamma_{ns}(G) = \gamma_c(G) = 2 \ \forall \ n \ge 3$ where $G = \overline{P_n}$.

Result 2.12: For any integer n ≥ 4

$\gamma(P_n) = \gamma'(P_n) = \gamma(\overline{P_n}) = \gamma'(\overline{P_n}) = 2$ iff n = 4, 5

Corollary 2.13: If G is a connected simple graph with $|V(G)| > 3$, D = {v} is the only minimum dominating set of G and $\gamma'(G) = |V| - 1$ then G is star graph.

Proof: Let G be any graph with $|V(G)| = n$.

Since D={v}is the only minimum dominating set of G, all vertices of G are connected with v, Also $\gamma'(G) = |V| - 1$ then the inverse dominating set of G consists all vertices of G other than V, therefore, no vertices of G – V are adjacent to each other which implies every vertices of G other, than v are pendent vertices ⇒ d (v) = n – 1 and $d(v_i) = 1$, for all $v_i \in G$ and $v_i \neq v$.

Hence G is a star graph.

Theorem 2.14: For any integer n ≥ 3.

(i) $\gamma(C_n) = \gamma'(C_n) = \gamma_i(C_n) = \gamma_s(C_n) = \left\lceil \frac{n}{3} \right\rceil$

(ii) $\gamma_g(C_n) = \left\lceil \frac{n}{3} \right\rceil$ if n ≥ 3 and n ≠ 5.

(iii) $\gamma_t(C_n) = \begin{cases} 2k+1 & \text{if } n = 4k+1 \\ 2\left\lceil \frac{n}{4} \right\rceil & \text{otherwise} \end{cases}$

(iv) $\gamma_p(C_n) = 2\left\lceil \frac{n}{4} \right\rceil$

(v) $\gamma_c(C_n) = \gamma_{ns}(C_n) = n - 2$

Proof: Let G = C_n be a cycle of length n and its vertices are denoted by $v_1, v_2, v_3, \ldots, v_n$, such that $v_i v_{i+1} \in E(G)$, $\forall$ i = 1 . . . $\overline{n-1}$ and $v_1 v_n \in E(G)$, we are going to prove this theorem in three cases.

Case (i): n = 3k, k = 1, 2, 3 . . . .

Choose, $D_1 = \{v_{3i+1} / i = 0, 1, 2, \ldots \text{ and } 3i + 1 \leq n\}$

$D_2 = \{v_{3i+2} / i = 0, 1, 2, \ldots; 3i + 2 \leq n\}$ and $D_3 = \{v_{3i+3} / i = 0, 1, 2, \ldots; 3i + 3 \leq n\}$

Now D_1, D_2 and D_3 are the minimum dominating sets of C_n also the elements of D_i, i = 1, 2, 3, . . . are independent and the induced subgraph <V – D> is disconnected, clearly each set D_1, D_2 and D_3 are mutually disjoint. Therefore, D_2 and D_3 are the inverse dominating set of D_1 and vise versa.

The cardinality of D_1, D_2 and D_3 is $\frac{n}{3} = \left\lceil \frac{n}{3} \right\rceil$ [□ n is a multiple of 3]

$\Rightarrow \gamma(C_n) = \gamma'(C_n) = \gamma_i(C_n) = \gamma_s(C_n) = \left\lceil \frac{n}{3} \right\rceil$ if n = 3k.

Case (ii): n = 3k + 1.

Now divide the graph G into k + 1 induced sub graphs G_i, i = 1, 2, . . , $\overline{k+1}$ containing three vertices,

$G_i = \{v_{3j+1}, v_{3j+2}, v_{3j+3} / i = 1, 2, \ldots k; j = i - 1\}$ and $G_{k+1} = \{v_n\}$

Let $D = \{v_{3i+2} \mid i = 0, 1, \ldots, k\} \cup G_{k+1}$ is the required minimum dominating set of G, all vertices of D are independent in G and the induced subgraph <V – D> is disconnected .

Now $|D| = k + 1$ (iii)

n = 3k + 1 [□ n = 3k + 1]

$\frac{n}{3} = k + \frac{1}{3} \Rightarrow \left\lceil \frac{n}{3} \right\rceil = k + \left\lceil \frac{1}{3} \right\rceil$

$= k + 1$

$$\Rightarrow \quad |D| = \left\lceil \frac{n}{3} \right\rceil \quad \text{[by (i)]}$$

Choose, $G_1=\{v_n, v_1, v_2\}$; $G_2 = \{v_3, v_4, v_5\}$; . . . ; $G_k = \{v_{n-4}, v_{n-3}, v_{n-2}\}$ and $G_{k+1} = \{v_{n-1}\}$ select $D' = \{v_{3i+1} / i = 0, 1, 2, . . . , k–1\} \cup G_{k+1}$ consists the elements $\{v_1, v_4, v_7, . . , v_{n-1}\}$ is the required inverse dominating set of C_n, by the same proof given in case (ii)

Therefore, $|D'| = \left\lceil \frac{n}{3} \right\rceil$

Case (iii): If n = 3k + 2,

Here also we divide the vertices of G as in case (ii)

$$G_i = \left\{v_{3j+1}, v_{3j+2}, v_{3j+3} / i = 1, 2, \ldots k;\ j = i - 1\right\} \text{ and } G_{k+1} = \{v_{n-1}, v_n\}$$

Choose the elements of D as $D = \{v_{3i+2}/ i = 0, 1, 2, . . ., k\}$ is the required minimum dominating set with cardinality k +1 and choose $D'=\{v_{3i+1} / i = 0, 1, 2, . . , k\}$ then $D'=\{v_1, v_4, v_7, .. , v_{n-1}\}$ is the required inverse dominating set with minimum cardinality k + 1.

$$n = 3k + 2 \Rightarrow \frac{n}{3} = k + \frac{2}{3}$$

$$\Rightarrow \left\lceil \frac{n}{3} \right\rceil = k + \left\lceil \frac{2}{3} \right\rceil$$

$$\Rightarrow \left\lceil \frac{n}{3} \right\rceil = k + 1$$

$$\Rightarrow |D| = \left\lceil \frac{n}{3} \right\rceil \quad [\square\ |D| = k + 1]$$

Also each set D and D′ are independent in C_n. In all cases the induced subgraph $<C_n – D>$ is disconnected.

Hence $\gamma(C_n) = \gamma'(C_n) = \gamma_i(C_n) = \gamma_s(C_n) = \left\lceil \frac{n}{3} \right\rceil$

If n = 4, then the graph and its complement are given as below

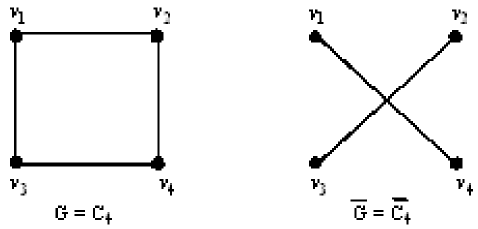

Figure: 1

$D = \{v_1, v_2\}$; $D' = \{v_1, v_2\}$

$\gamma(C_4) = \gamma(\overline{C_4}) = 2$

Hence, $\gamma_g(C_4) = 2$

If n = 5, the graph and its complement are represented as below.

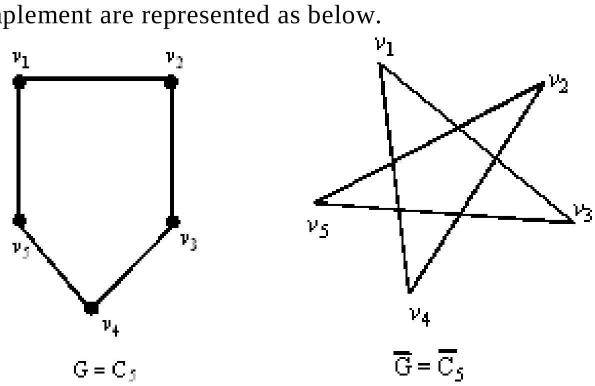

Figure: 2

In G,

$D_1 = \{v_1, v_3\}$; $D_2 = \{v_1, v_4\}$; $D_3 = \{v_2. v_4\}$
$D_4 = \{v_2, v_5\}$; $D_5 = \{v_3, v_5\}$ are the minimum dominating sets of G.

In $\overline{G}$,

$\overline{D_1} = \{v_1, v_2\}$; $\overline{D_2} = \{v_1, v_5\}]$; $\overline{D_3} = \{v_2, v_3\}$
$\overline{D_4} = \{v_3, v_4\}$; $\overline{D_5} = \{v_4, v_5\}$ are the minimum dominating sets.

Clearly, none of the dominating set of G with cardinality two is a dominating set of $\overline{G}$.

$\Rightarrow \gamma_g(C_5) \neq 2$
$\Rightarrow \gamma_g(C_5) = 3$

In C_n, n > 5 all minimum dominating sets of C_n in also a dominating set of $\overline{C_n}$

Hence, $\gamma_g(C_n) = \left\lceil \frac{n}{3} \right\rceil \ \forall\ n > 3$ and $n \neq 5$.

(iii) Case (i) if n = 4k + 1. Let $v_1, v_2, v_3 \ldots, v_n$ be the vertices of the graph $G = C_n$ now divide G into k induced subgraph G_i, i = 1, . . . , k such that $G_{i+1} = \{v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+3}, v_{4i+4} / i = 0, 1, 2, \ldots, \overline{k-2}\}$ and the last partition G_k contains five vertices as $G_k = \{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$

Choose, $D_t = \{v_{4i+2,\ 4i+3} / i = 0, 1, \ldots, \overline{k-2}\} \cup \{v_{n-3}, v_{n-2}, v_{n-1}\}$ is the required minimum total dominating set with cardinality

$| D_t (G)| = (K - 1)\, 2 + 3$
$= 2k + 1$ (iv)

If $n \neq 4k + 1$. by case (i) of (iii)

$G_{i+1} = \{v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}, / i = 0, 1, 2, \ldots \overline{k-1}\}$ then the last partition

$$G_{k+1} = \begin{cases} \{v_{n-1}, v_n\} & \text{if } n = 4k + 2 \\ \{v_{n-2}, v_{n-1}, v_n\} & \text{if } n = 4k + 3 \end{cases}$$

then $D_t(G) = \{v_{4i+2}, v_{4i+3} / i = 0, 1, 2 \ldots, \overline{k-1}\} \cup \{\text{any two elements of } G_{k+1}\}$

Now $D_t(G)$ is the minimum total dominating set with cardinality

$| D_t (G)| = 2 (k + 1)$ (v)

$n = 4k + 2$ and $n = 4k + 3$

$\frac{n}{4} = k + \frac{1}{2}$ and $\frac{n}{4} = k + \frac{3}{4}$

$\left\lceil \frac{n}{4} \right\rceil = k + \left\lceil \frac{1}{2} \right\rceil$ and $\left\lceil \frac{n}{4} \right\rceil = k + \left\lceil \frac{3}{4} \right\rceil$

$\left\lceil \frac{n}{4} \right\rceil = k + 1$ and $\left\lceil \frac{n}{4} \right\rceil = k + 1$

$\Rightarrow 2. \left\lceil \frac{n}{4} \right\rceil = 2 (k + 1)$ and $2 \left\lceil \frac{n}{4} \right\rceil = 2 (k + 1)$

$\Rightarrow | D_t (G) | = 2 \left\lceil \frac{n}{4} \right\rceil$ if $n = \{4k + 2 \text{ and } 4k + 3\}$ [□ by (v)]

$$\text{Hence, } \gamma_t(G) = \begin{cases} 2k + 1 & \text{if } n = 4k + 1 \\ 2 \left\lceil \frac{n}{4} \right\rceil & \text{otherwise} \end{cases}$$

iv) The same argument given in (iii)

Let $T = \{v_{4i+2}, v_{4i+3} / i = 0, 1, 2 \ldots . \overline{k-1}\}$ then $D_p(G)$ $T \cup \{v_{n-1}, v_n\}$ is the required minimum dominating set of G and $|Dp(G)| = 2(k+1)$ (vi)

If, $n = 4k + 1$

$$\frac{n}{4} = k + \frac{1}{4} \Rightarrow \left\lceil \frac{n}{4} \right\rceil = k + \left\lceil \frac{1}{4} \right\rceil$$

$$\Rightarrow \left\lceil \frac{n}{4} \right\rceil = k + 1$$

$$\Rightarrow 2\left\lceil \frac{n}{4} \right\rceil = 2(k+1)$$

$$\Rightarrow |D_p(G)| = 2\left\lceil \frac{n}{4} \right\rceil \qquad [\square \text{ by } 1]$$

if $n = 4k + 2$ and if $n = 4k + 3$

$$\frac{n}{4} = k + \frac{2}{4} \quad \text{and} \quad \frac{n}{4} = k + \frac{3}{4}$$

$$\left\lceil \frac{n}{4} \right\rceil = k + \left\lceil \frac{2}{4} \right\rceil \quad \text{and} \quad \left\lceil \frac{n}{4} \right\rceil = k + \left\lceil \frac{3}{4} \right\rceil$$

$$= k + 1 \quad \text{and} \quad = k + 1$$

$$\Rightarrow 2(k+1) = 2\left\lceil \frac{n}{4} \right\rceil \quad \text{and} \quad \Rightarrow 2(k+1) = 2\left\lceil \frac{n}{4} \right\rceil$$

$$\Rightarrow |D_p(G)| = 2\left\lceil \frac{n}{4} \right\rceil \quad \text{and} \quad |D_p(G)| = 2\left\lceil \frac{n}{4} \right\rceil$$

Hence $\gamma_p(G) = 2\left\lceil \frac{n}{4} \right\rceil \ \forall\, n > 3$.

v) Let $T = \{v_i, v_{i+1} / v_i v_{i+1} \in E(G)\}$ the $V - T$ and $V - \{v_1 . v_n\}$ is a connected and non split dominating sets of G.

Therefore, $\gamma_c(C_n) = \gamma_{ns}(C_n) = n - 2 \ \forall\, n > 3$.

Hence the proof.

Corollary 2.15: If G is one corona (k_n o k_1) then

$$\gamma(G) = \gamma'(G) = \gamma_T(G) = \gamma_i(G) = \gamma_c(G) = \gamma_s(G) = \gamma_{ns}(G) = \gamma_g(G) = n.$$

$$\gamma_p(G) = \begin{cases} n & \text{if } n = 2k \\ n+1 & \text{if } n = 2k+1 \end{cases}$$

Proof: Let the vertex set of $G = K_n$ o K_1 is represented in figure 1.

Let $S_1 = \{v_1, v_2, v_3, \ldots, v_n\}$ and $S_2 = \{u_1, u_2, u_3, \ldots, u_n\}$

Then $D = S_1$; $D' = S_2$

$D_T = S_1$; $D_i = S_2$

$D_C = S_1$; $D_{ns} = S_2$

$D_p = \{(v_{2i+1}, v_{2i+2}) / i = 0, 1, 2 \ldots k - 1\}$

Therefore,

$$\gamma(G) = \gamma'(G) = \gamma'_t(G) = \gamma_i(G) = \gamma_c(G) = \gamma_s(G) = \gamma_{ns}(G) = \gamma_g(G) = n$$

If $n = 2k + 1$

$Dp = \{v_{2i+1} . v_{2i+2}\} \cup \{v_{n-1}, v_n\} / i = 0, 1, \ldots, \overline{k-1}$ and $2i + 2 \leq 2k\}$ is the required minimum paired dominating set of G,

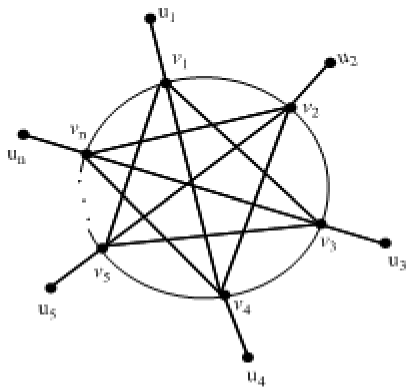

Figure: 3

$$
\begin{aligned}
| D_p(G) | &= 2k + 2 \\
&= 2k + 1 + 1 \\
&= n + 1 \quad [\because 2k + 1 = n]
\end{aligned}
$$

Therefore, $\gamma_p(G) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \quad \forall\ n \geq 3 \end{cases}$

Corollary 2.16: For any graph G is the r corona of $K_n, \forall\ n \geq 3$

$\gamma(G) = \gamma'(G) = \gamma_t(G) = \gamma_s(G) = \gamma_{ns}(G) = \gamma_i(G) = n$ and

$\gamma_p(G) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$

Theorem 2.17: Let G be a barbeled graph then

$\gamma(G) = \gamma'(G) = \gamma_t(G) = \gamma_i(G) = \gamma_s(G) = \gamma_{ns}(G) = \gamma_p(G) = 2$

Proof: The Barbelled graph G is given in figure as below.

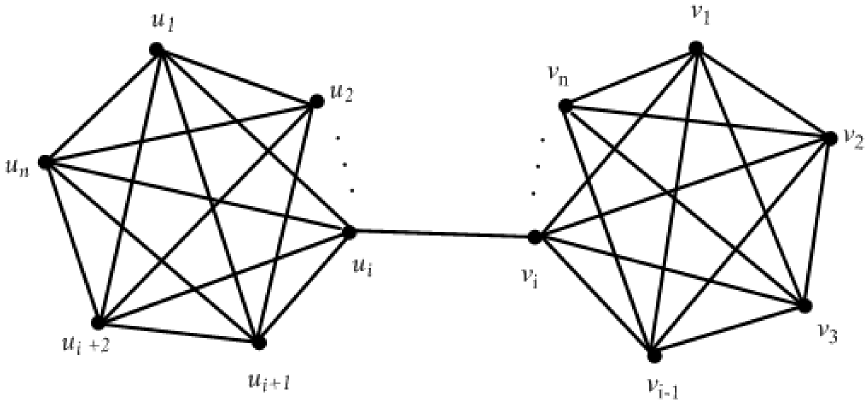

Figure: 4

Let $S_1 = \{u_1, u_2, u_3 \ldots u_{i-1}, u_i, u_{i+1}, \ldots, u_n\}$

$S_2 = \{v_1, v_2, v_3, ., \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n\}$

$N(u_j) = S_1$; $j \neq i$ and $N(u_i) = S_1 \cup \{v_i\}$; $N(v_j) = S_2$, $j \neq i$ and $N(v_j) = S_2 \cup \{u_i\}$

$N(v_j) = S_2$, $j \neq i$ and $N(v_j) = S_2 \cup \{u_i\}$

Now $D(G) = \{u_i, v_i\} \Rightarrow D'(G) = \{(u_j, v_j);\ j = 1, \ldots, n;\ j \neq i\}$

$D_t(G) = \{u_i, v_i\}$; $Di(G) = \{(u_j, v_j);\ j = 1, 2, \ldots;\ j \neq i\}$

$D_S(G) = \{(u_i, v_i)$; $D_{ns}(G) = \{u_j, v_j\}$, $j = 1 \ldots, n$, $j \neq i$

$D_k(G) = \{u_i, v_i\}$ for all i= 1,2,…,n

The vertex u_i is connected with vertex $v_i \in S_2$ in G, therefore u_i is connected with all the vertices of S_2 other than v_i in $\overline{G}$. Similarly v_i is connected with $u_i \in S_1$ in G. Therefore, v_i is connected with all the vertices of S_1 other than u_i in G. Hence

$D_g(G) = \{u_i, v_i\}$ and $D_p(G) = \{u_i, v_i\}$

$\gamma(G) = \gamma'(G) = \gamma_i(G) = \gamma_s(G) = \gamma_{ns}(G) = \gamma_c(G) = \gamma_g(G) = \gamma_p(G) = 2.$

Theorem 2.18: Let G be any complete bipartite graph with m, n vertices then

$\gamma(G) = \gamma'(G) = \gamma_t(G) = \gamma_c(G) = \gamma_{ns}(G) = \gamma_g(G) = \gamma_p(G) = 2$ and

$\gamma_i(G) = \gamma_s(G) = \{$m or n whichever in less$\}$

Proof: The vertices of $G = K_{m,n}$ are partitioned in to two sets S_1 and S_2 such that

$S_1 = \{u_1, u_2, \ldots, u_n\}$ and $S_2 = \{v_1, v_2, \ldots, v_n\}$, Since $G = K_{m,n}$

$N(u_i) = S_2 \ \forall\ i = 1, 2, \ldots, m$ and $N(v_i) = S_1 \ \forall\ i = 1, 2, \ldots, n$

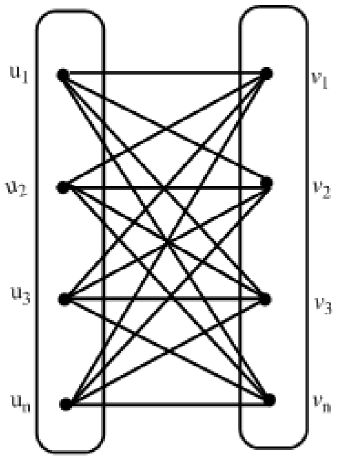

Figure: 5

Therefore, each pair of vertices $\{(u_i, v_j)/\ u_i \in S_1\ v_j \in S_2\}$ is a dominating set of G. $u_i v_j \in E(G)$ which implies that

$$D(G) = D_T(G) = D_c(G) = D_p(G) = D_g(G) = \left\{ (u_i, v_j) \middle/ u_i \in S_1 \text{ and } v_j \in S_2 \right\}$$

Also, any induced subgraph <V – D> is connected.

Therefore,

$D_{nS}(G) = \{(u_i, v_j)/\ u_i \in S_1;\ vj \in S_2,\ i \neq j\}$. For any dominating set

$D = \{(u_i, v_j)\ /\ u_i \in S_1;\ v_j \in S_2\}$

$D' = \{(u_j, v_i)/\ u_j \in S_1,\ v_i \in S_2,\ i \neq j\}$ is the inverse dominating set of D.

Since, u_i is not adjacent with any element of S_1 in G, u_i, $i = 1 \ldots, m$ is adjacent with all the vertices of S_1 in $\overline{G}$,

Similarly v_j, $j = 1, \ldots, n$ is connected with all elements of S_2 in $\overline{G}$ which implies (u_i, v_j) is a dominating set of $\overline{G}$, that is, $D_g(G) = \{(u_i, v_j)/\ u_i \in S_1$ and $v_j \in S_2\}$

Clearly $+V(G) - S_1$, and $+V(G) - S_2$, are disconnected. Also the elements of S_1 and s_2 are independent, hence independent and split dominating set of G is either S_1 or S_2 which having the minimum numbers of vertices.

$$\text{That is, } \gamma_i(G) = \gamma s(G) = \begin{cases} m, & \text{if } m < n \\ n & \text{otherwise} \end{cases}$$

Corollary 2.19: If $G = K_{m,n}$ then $\overline{G}$ is a disconnected graph with two components and each components is a complete subgraph with m, n vertices.

Hence

$\gamma(\overline{G}) = \gamma'(\overline{G}) = \gamma_i(\overline{G}) = \gamma_s(\overline{G}) = \gamma_g(\overline{G}) = 2$ and $\gamma_t(\overline{G}) = \gamma_p(\overline{G}) = 4.$

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