International Journal of Mathematical Archive-2(8), 2011, Page: 1310-1313
Available online through www.ijma.info ISSN 2229 – 5046

EXISTENCE OF COUPLED SOLUTION FOR MIXED MONOTONE OPERATORS

Dr. K. L. Bondar, and A. B. Jadhav*

P .G. Dept. of Mathematics, N. E. S. Science College, Nanded – 431 605 (M.S.), India
Dept. of Mathematics, D. S. M. College, Parbhani (M.S.), India

E-mail: klbondar_75@rediffmail.com

(Received on: 29-07-11; Accepted on: 11-08-11)

ABSTRACT

In this paper we obtain some new and general existence theorems of coupled solution for mixed monotone operators with perturbation. These results generalizes results of D. Gau and V. Laxmikantham [4].

Keywords: Existence of solution, Fixed Point Theorem, Coupled Solution, Cone.

---

1. INTRODUCTION:

Various existence and uniqueness theorems of fixed points for monotone operators are of great use in the study of nonlinear equations [1-6, 9, 10]. In 1996, Zang [8] investigated the existence and uniqueness of positive fixed point for concave and convex mixed monotone operators. Ke Li, Jian Liang and Ti-Jun Xiao [6] obtained some new and general existence and uniqueness theorems of positive fixed points for mixed monotone operators with perturbation. He also gave some applications to nonlinear integral equations on unbounded region. In this paper we study the existence of coupled solution of equation

$$A(x, y) + B(x) = x, \tag{1}$$

where A is mixed monotone operator and completely continuous and $x\in E$ where E is a Banach space.

Throughout this paper, E is a Banach space with norm $\|.\|$, and P is a cone in E. So a partial order in E is given by $x \geq y$ if $y - x \in P$.

Recall that a cone P is said to be normal if there exists a constant N such that

$$0 \leq x \leq y \Rightarrow \|x\| \leq N \| y \|,$$

where N is called a normal constant.

P is said to be solid if interior of P is nonempty.

Definition: 1.1 Let $W \subset E$. Operator $A : W \times W \rightarrow E$ is solid to be mixed if $A(x, y)$ is increasing in x and decreasing in y, *i.e.* $x_1, x_2, y_1, y_2 \in W$; $x_1 \leq x_2$, $y_1 \geq y_2 \Rightarrow A(x_1, y_1) \leq A(x_2, y_2)$.

Definition: 1.2 A point $x^* \in W$ is called a fixed point of A if $A(x^*,x^*) = x^*$. A point $(x^*, y^*) \in W \times W$ is said to be a coupled fixed point of A if $A(x^*,y^*) = x^*$ and $A(y^*,x^*) = y^*$.

Definition: 1.3 A point (x^*, y^*) is said to be a coupled solution of (1) if

$$A(x^*, y^*) + B(x^*) = x^* \text{ and } A(y^*, x^*) = y^*.$$

Definition: 1.4 An operator $A : W \times W \rightarrow E$ is said to be completely continuous if it is continuous and compact.

2. MAIN RESULTS

Theorem: 2.1 Let P be normal cone with normal constant N. A: $[u, v] \times [u, v] \rightarrow E$ be a mixed monotone and completely continuous such that

$$A(u, v) \geq u, \;\; A(v, u) \leq v, \; Bu \geq 0, \; Bv \leq 0.$$

***Corresponding author:** *Dr. K. L. Bondar**, ***E-mail:** *klbondar_75@rediffmail.com*

Suppose $(I-B)^{-1}$ exists, continuous and nondecreasing. Then the equation (1) has a coupled solution (u^*, v^*) which is minimal and maximal in the sense that, if $(\bar{x}, \bar{y})$ is any coupled solution of (1) then $u^* \leq \bar{x} \leq v^*$ and $u^* \leq \bar{y} \leq v^*$.

Proof: Given equation is

$$A(x, y) + B(x) = x$$

$$A(x, y) = (I-B)(x)$$

$$(I-B)^{-1} A(x, y) = x. \tag{2}$$

If we let $C = (I-B)^{-1} A$ then above equation becomes

$$C(x, y) = x. \tag{3}$$

Since $(I-B)^{-1}$ is nondecreasing and A is mixed monotone, therefore C is mixed monotone. Now

$$u \leq A(u, v)$$

$$u \leq A(u, v) + B(u)$$

$$(I-B)\, u \leq A(u, v)$$

$$u \leq (I-B)^{-1} A(u, v)$$

$$u \leq C(u, v).$$

Similarly

$$A(v, u) \leq v$$

$$A(v, u) + B(v) \leq v$$

$$A(v, u) \leq (I-B)\, v$$

$$(I-B)^{-1} A(v, u) \leq v$$

$$C(\mathrm{v}, u) \leq v.$$

Thus A: $[u, v] \to [u, v]$.

Now define the sequences

$$u_n = C(u_{n-1}, v_{n-1}),\ v_n = C(v_{n-1}, u_{n-1}),\ (n = 1, 2, \ldots.) \tag{4}$$

From the hypothesis it is clear that

$$u_0 \leq u_1 \leq v_1 \leq v_0.$$

Suppose $u_{n-1} \leq u_n \leq v_n \leq v_{n-1}$. Then

$$u_n = C(u_{n-1}, v_{n-1}) \leq C(u_n, v_{n-1}) \leq C(u_n, v_n) = u_{n+1}$$

$$v_n = C(v_{n-1}, u_{n-1}) \geq C(v_n, u_{n-1}) \geq C(v_n, u_n) = v_{n+1}.$$

Therefore

$$u_{n+1} = C(u_n, v_n) \leq C(v_n, u_n) = v_{n+1}.$$

Hence by induction we have

$$u = u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0 = v.$$

Since A is completely continuous and $(I-B)^{-1}$ is continuous, C is completely continuous. Hence there exists subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$ of $\{u_n\}$ and $\{v_n\}$ respectively such that,

$$u_{n_k} \to u^*, \quad v_{n_k} \to v^*, \quad u^*, v^* \in [u, v].$$

Moreover $u_{n_k} \le v_{n_k}$, therefore $u^* \le v^*$. For $m > n_k$, $u_{n_k} \le u_m \le v_m \le v_{n_k}$.

$$u^* - u_m \le u^* - u_{n_k}, \; v^* - v_m \le v^* - v_{n_k}.$$

By virtue of normality of *P,*

$$\| u^* - u_m \| \le N \| u^* - u_{n_k} \|$$

and

$$\| v^* - v_m \| \le N \| v^* - v_{n_k} \|.$$

Thus $u_m \to u^*$ and $v_m \to v^*$ $(m \to \infty)$.

Taking limit as $n \to \infty$ on both sides of (4), we get

$$u^* = C(u^*, v^*), \; v^* = C(v^*, u^*).$$

Since C is continuous, therefore (1) has a coupled solution (u^*, v^*). i. e.

$$A(u^*, v^*) + B(u^*) = u^*, \; A(v^*, u^*) + B(v^*) = v^*.$$

To prove the minimal and maximal property of (u^*, v^*), assume $\bar{x}, \bar{y}, \in [u_0, v_0] \times [u_0, v_0]$ is any coupled solution of (1).

Since $u_0 \le \bar{x} \le v_0$, $u_0 \le \bar{y} \le v_0$,

$$u_1 = A(u_0, v_0) \le A(\bar{x}, \bar{y}) = \bar{x} \le A(v_0, \bar{y}) \le A(v_0, u_0) = v_1,$$

$$u_1 = A(u_0, v_0) \le A(\bar{y}, \bar{x}) = \bar{y} \le A(v_0, \bar{x}) \le A(v_0, u_0) = v_1.$$

In general we obtain

$$u_n \le \bar{x} \le v_n, \; u_n \le \bar{y} \le v_n.$$

Taking limit as $n \to \infty$, we get

$$u^* \le \bar{x} \le v^*, \; u^* \le \bar{y} \le v^*.$$

Hence the theorem is proved.

If $B \equiv 0$, then Theorem 2.1, reduces to Theorem 2.1.7 in [4].

Corollary: 2.2 [4] Let P be normal with normal constant N, $A : [u_0, v_0] \times [u_0, v_0] \to E$ be a mixed monotone operator and completely continuous such that

$$A(u_0, v_0) \ge u_0, \quad A(v_0, u_0) \le v_0.$$

Then

$$A(x, y) = x \tag{5}$$

has a coupled solution (u^*, y^*) which is minimal and maximal in the sense that, if $(\bar{x}, \bar{y})$ is any coupled solution of (5) then $u^* \le \bar{x} \le v^*$ and $u^* \le \bar{y} \le v^*$.

Theorem: 2.3 Let P be normal cone with normal constant N. S: $[u, v] \to [u, v]$ be a nondecreasing and completely continuous mapping. Suppose B: $[u, v] \to [u, v]$ such that $(I - B)^{-1}$ exists, continuous and nondecreasing. Moreover assume that,

$$A(u) \ge u, \; A(v) \le v, \; B(u) \ge 0, B(v) \le 0.$$

Then the equation

$$Sx + Bx = x \tag{6}$$

has a minimal and a maximal solution in $[u, v]$.

Proof: Set a mapping $A : E \times E \to E$ such that

$$A(x, y) = S(x), \; x, y \in [u, v].$$

Clearly A is mixed monotone operator which is completely continuous. Moreover coupled solutions of (1) are the solutions of (6). Now result follows from the Theorem 2.1.

If we assume $B \equiv 0$ in above theorem, then it reduces to Theorem 2.1.1 in [4].

Corollary: 2.4 [4] Let P be a normal cone with normal constant N. Suppose $A: [u, v] \times [u, v] \rightarrow [u, v]$ be nondecreasing and completely continuous such that

$$u \leq A(u), \; A(v) \leq v, \; 0 \leq B(u), \; B(v) \geq 0.$$

Then $A(x, y) = x$ has a minimal and a maximal solution in $[u, v]$.

3. REFERENCES:

[1] K. Demling, Nonlinear Functional Analysis, *Springer-Verlag, New York* (1985).

[2] Y. H. Du, Fixed points of increasing operators in ordered Banach spaces and applications *Appl, Anal.,* **38** (1990), 1-20.

[3] D. J. Gao, Fixed points of mixed monotone operators with applications, *Appl. Anal.,* **31** (1988), 215-224.

[4] D. J. Gao, V. Laxmikantham, Nonlinear problems in abstracts cones, *Academic Press, Boston,* (1998).

[5] K. Li, J. Liang, T. Xiao, A fixed point theorem for convex and decreasing operators, *Nonlinear Anal.,* **63** (2005), 209-216.

[6] K. Li, J. Liang, T. Xiao, New existence and uniqueness theorems of positive fixed points for mixed monotone operators with perturbation, *J. Math. Anal. Appl.* **328** (2007), 753-766.

[7] W. Rudin, Functional Analysis, *McGraw-Hill, New York* (1991).

[8] Z. Zang, New fixed point theorems of mixed monotone operators and applications, *J. Math. Anal. Appl.,* **204** (1996), 307-319.

****

