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DIRECT PRODUCT OF (Q, L)-FUZZY SUBGROUPS AND THEIR PROPERTIES

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ABSTRACT

In this paper, some properties of (Q, L)-fuzzy subgroup of a group are discussed, and obtained some algebraic properties on the direct product of (Q, L)-fuzzy subgroups by means of Q-level sets.

Keywords: *(Q, L)-fuzzy subset, (Q, L)-fuzzy subgroups, (Q, L) – fuzzy normal subgroup, Q-level subsets.*

SECTION 1 – INTRODUCTION

The concept of fuzzy set was introduced by Zadeh [7]. Rosenfield [6] gave the idea of subgroups. Solairajuand Nagarajan [4, 5] introduced and defined a new algebraic structure of Q-fuzzy groups. Asokkumer Ray [1] defined a product of fuzzy groups. Goguen [2] studied the fuzzy set theory by studying L-fuzzysets. In this paper, we discuss some equivalent characterizations of direct product of (Q, L)-fuzzy groups by means of Q-level subsets.

SECTION 2 – BASIC DEFINITIONS

Definition 2.1: Let X and Q be any two non-empty sets. A mapping $\mu: X \times Q \rightarrow [0, 1]$ is called a Q-fuzzy set in X.

Definition 2.2: Let X be a non-empty set and L = (L, ≤) be a lattice with least element 0 and greatest element 1 and Q be a non-empty set .A (Q, L)-fuzzy subset A of X is a function $A: X \times Q \rightarrow L$.

Definition 2.3: A (Q, L) - fuzzy subset λ of G is said to be a (Q, L)-fuzzy subgroup of G if for all $x, y \epsilon G \ and \ q \in Q$
(i) $(xy, q) \geq \lambda(x, q) \wedge \lambda(y, q)$
(ii) $\lambda(x^{-1}, q) = \lambda(x, q)$

SECTION 3 – PROPERTIES ON (Q, L) – FUZZY SUBGROUP

Theorem 3.1: A (Q, L)-fuzzy subset λ of G is a (Q, L)-fuzzy subgroup of G if and only if
$$(xy^{-1}, q) \geq \lambda(x, q) \wedge \lambda(y, q), \forall x, y \in G \ and \ q \in Q.$$

Proof: λ is a (Q, L)-fuzzy subgroup of G.
$\Leftrightarrow \lambda(xy, q) \geq \lambda(x, q) \wedge \lambda(y, q) \ and \ \lambda(x^{-1}, q) = \lambda(x, q)$

$\Leftrightarrow \lambda(xy^{-1}, q) \geq \lambda(x, q) \wedge \lambda(y, q), \forall x, y \in G \ and \ q \in Q$

Definition 3.2: Let A be a (Q, L)-fuzzy subset of G. For $\alpha \in L$, a Q-level subset of A corresponding to α is the set $A_\alpha = \{x \in G, q \epsilon Q: A(x, q) \geq \alpha\}$

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Theorem 3.3: If A is a (Q, L)–fuzzy subset of a group G. Then A is a (Q, L)–fuzzy subgroup of G if and only if A_α is a subgroup of a group G for all $\alpha \in L$.

Proof: Let $x, y \in G, q \in Q$

$A(xy^{-1}, q) \geq A(x, q) \wedge A(y, q)$

$A(xy^{-1}, q) \geq \alpha$

$\Rightarrow xy^{-1} \in A_\alpha \Rightarrow A_\alpha$ is a subgroup of a group G for all $\alpha \in L$.

Definition 3.4: A (Q, L) - fuzzy subgroup A of group G is a (Q, L)–fuzzy normal subgroup of G

$\mu_A(xyx^{-1}, q) = \mu_A(y, q)$ or $\mu_A(xy, q) \geq \mu_A(yx, q)$ $for\ all\ x, y \in G\ and\ q \in Q$.

Theorem 3.5: If A is a (Q, L) – fuzzy subset of a group G. Then A is a (Q, L) – fuzzy normal subgroup of G if and only if A_α is a normal subgroup of a group G for all $\alpha \in L$.

Proof: Let A be a (Q, L)–fuzzy normal subgroup of a group G and the level subset A_α,$\alpha \in L$ isa subgroup of G.Let $x \in G$ and $a \in A_\alpha$. Then $\mu_A(xax^{-1}, q) = \mu_A(a, q) \geq \alpha$. Hence A_α is a normal subgroup of a group G for all $\alpha \in L$.

Definition 3.6: Let A and B be two (Q, L) –fuzzy subgroups of G. Then A and B are said to be (Q, L) –fuzzy conjugate subgroup of G if for some $\in G$, $\mu_A(x, q) = \mu_B(g^{-1}xg, q), \forall x \in G$.

Definition 3.7: Let A be a (Q, L) –fuzzy subset in a set S, the strongest(Q, L) fuzzy relation on S, that is (Q, L) –fuzzy relation on A is V given by $\mu_V\big((x, y), q\big) = \mu_A(x, q) \wedge \mu_A(y, q)\ \forall x, y \in S$.

SECTION 4: DIRECT PRODUCT OF (Q, L)-FUZZY SUBGROUPS

Definition 4.1: Let A and B be two (Q, L)-fuzzy subsets of X and Y respectively. Then the Cartesian product of A and B is denoted by $A \times B$ and is defined as

$$A \times B = \{< \big((x, y), q\big), \mu_{A\times B}((x, y), q) >: x \in X, y \in Y\ and\ q \in Q\}$$

where $\mu_{A\times B}((x, y), q) = \mu_A(x, q) \wedge \mu_B(y, q)$.

Theorem 4.2: If A and B be two (Q, L) – fuzzy subsets of X and Y respectively, then $(A \times B)_\alpha = A_\alpha \times B_\alpha$for $\alpha \in L$.

Proof: Let $(x, y)\, \epsilon (\, A \times B)_\alpha\ and\ q\epsilon Q$.

Then $\mu_{A\times B}\big((x, y), q\big) \geq \alpha$

$\Leftrightarrow \mu_A(x, q) \wedge \mu_B(y, q) \geq \alpha$

$\Leftrightarrow \mu_A(x, q) \geq \alpha, \mu_B(y, q) \geq \alpha$

$\Leftrightarrow x\epsilon A_\alpha\ , y\epsilon B_\alpha$

$\Leftrightarrow (x, y)\ \epsilon\ A_\alpha \times B_\alpha$ for$\alpha \in L$

Hence,$(\, A \times B)_\alpha = A_\alpha \times B_\alpha$, for $\alpha \in L$.

Theorem4.3: Let A and B be two (Q, L) – fuzzy subgroups of group G_1 and G_2 respectively.Then $A \times B$ is a (Q, L) – fuzzy subgroup of group $G_1 \times G_2$.

Proof: Since A and B are (Q, L) – fuzzy subgroups of group G_1 and G_2 respectively. Then $A_\alpha\ and\ B_\alpha$ are subgroups of group G_1 and G_2 respectively.

$\Rightarrow A_\alpha \times B_\alpha$ is asubgroup of $G_1 \times G_2$, for $\alpha \in L$.

$\Rightarrow (A \times B)_\alpha$ is asubgroup of$G_1 \times G_2$,for $\alpha \in L$. (By thm 2.6)

$\Rightarrow A \times B$is a (Q,L) – fuzzy subgroup of group $G_1 \times G_2$.

Theorem 4.4: Let A and B be two (Q, L) – fuzzy normal subgroups of group G_1 and G_2 respectively. Then $A \times B$ is a (Q, L) – fuzzy normal subgroup of group $G_1 \times G_2$

Proof: Since A and B are (Q, L) – fuzzy normal subgroups of group G_1 and G_2 respectively. Then $A_\alpha\ and\ B_\alpha$ are normal subgroups of group G_1 and G_2 respectively.

$\Rightarrow A_\alpha \times B_\alpha$ *is* a normal subgroup of $G_1 \times G_2$, for $\alpha \in L$.

$\Rightarrow (A \times B)_\alpha$ *is* a normal subgroup of $G_1 \times G_2$,for $\alpha \in L$. (By thm 2.6)

$\Rightarrow A \times B$ is a (Q, L) – fuzzy normal subgroup of group $G_1 \times G_2$.

Remark 4.5: Let A and B be (Q, L) – fuzzy subgroups of group G_1 and G_2 respectively. If $A \times B$ is a (Q, L) –fuzzy subgroup of group $G_1 \times G_2$, then it is not necessary that both A and B should be (Q, L) – fuzzy subgroup of group $G_1 \times G_2$.

Example 4.6: Let $G_1 = \{e_1, x\}$ where $x^2 = e_1$, $G_2 = \{e_2, a, b, ab\}$ $where\ a^2 = b^2 = e_2\ and\ ab = ba$.

Then $G_1 \times G_2 = \{(e_1, e_2), (e_1, a), (e_1, b), (e_1, ab), (x, e_2), (x, a), (x, b), (x, ab)\}$

Let $A = \{< (e_1, q), (0.5, q) >, < (x, q), (0.8, q) >\}$ and $B = \{< (e_2, q), (0.7, q) >, < (a, q), (1, q) >, <$ *b,q,0.8, q>,<(ab,q),0.7,q>* be (Q, L)-fuzzy subsets of *G1*and *G2* respectively.

Then
$A \times B = \{< ((e_1, e_2), q), (0.5, q) >, < ((e_1, a), q), (0.5, q) >, < ((e_1, b), q), (0.5, q) >, < ((e_1, ab), q), (0.5, q) >$ *,<x,e2,q,0.7,q>,<x,a,q,0.8,q>, <x,b,q,0.8,q>,<x,ab,q,0.7,q>.*

Here $A \times B$ is a (Q, L)-fuzzy subgroup of $G_1 \times G_2$ where A is a (Q, L)-fuzzy subgroup of G_1 but B is not a (Q, L)-fuzzy subgroup of G_2.

Theorem 4.7: Let $A\ and\ B$ be (Q, L)- fuzzy subgroups of $G_1$and G_2 respectively. Suppose that $e_1\ and\ e_2$ are the identity element of $G_1 and\ G_2$ respectively. If $A \times B$ is a (Q,L)-Fuzzy subgroup of $G_1 \times G_2$, then at least one of the two statements must holds.
(i) $\mu_B(e_2, q) \geq \mu_A(x, q)$for all $x \epsilon G_1 (ii)$ $\mu_A(e_1, q) \geq \mu_B(y, q)$ for all $y\ \epsilon G_2$

Proof: Let $A \times B$ is a (Q, L)-Fuzzy subgroup of $G_1 \times G_2$.

Suppose that (i) and (ii) does not holds.

Then we can find some $x \epsilon G_1\ and\ y\ \epsilon G_2$ such that $\mu_A(x, q) > \mu_B(e_2, q)$ and $\mu_A(e_1, q) < \mu_B(y, q)$.

Now $\mu_{A\times B}((x, y), q) = \mu_A(x, q) \wedge \mu_B(y, q) > \mu_B(e_2, q) \wedge \mu_A(e_1, q) = \mu_{A\times B}((e_1, e_2), q)$.

which implies that $A \times B$ is not a (Q, L)-Fuzzy subgroup of $G_1 \times G_2$,which is a contradiction.

Hence either $\mu_B(e_2, q) \geq \mu_A(x, q)$ for all $x \epsilon G_1$, $q\ \epsilon Q$ or$\mu_A(e_1, q) \geq \mu_B(y, q)$ for all $y\ \epsilon G_2$, $q\ \epsilon Q$.

Theorem 4.8: Let $A\ and\ B$ be (Q, L)-fuzzy subsets of $G_1$and G_2 respectively such that $\mu_A(x, q) \leq \mu_B(e_2, q), x \in G_1$,$e_2$be the identity element of G_2, $q \in Q$.If $A \times B$ is a (Q, L)-fuzzy subgroup of $G_1 \times G_2$, then A is a (Q, L)-fuzzy subgroup of G_1.

Proof: Let $x, y \in G_1$. Then $(x, e_2), (y, e_2) \in G_1 \times G_2$.

Since $\mu_A(x, q) \leq \mu_B(e_2, q), for\ all\ x \in G_1$,$e_2 \in G_2$,$q \in Q$.

$$\begin{aligned}
\mu_A(xy^{-1}, q) &= \mu_A(xy^{-1}, q) \wedge \mu_B(e_2e_2, q) \\
&= \mu_{A\times B}((xy^{-1}, e_2e_2), q) = \mu_{A\times B}((x, e_2)(y^{-1}, e_2)), q) \\
&\geq \mu_{A\times B}((x, e_2), q) \wedge \mu_{A\times B}((y^{-1}, e_2), q) (\because A \times B \text{ is a (Q, L)-Fuzzy subgroup of } G_1 \times G_2) \\
&= (\mu_A(x, q) \wedge \mu_B(e_2, q)) \wedge ((\mu_A(y^{-1}, q) \wedge \mu_B(e_2, q)) \\
&= \mu_A(x, q) \wedge \mu_A(y^{-1}, q) \\
&\geq \mu_A(x, q) \wedge \mu_A(y, q)
\end{aligned}$$

Hence A is an (Q, L)-fuzzy subgroup of G_1.

Corollary 4.9: Let $A\ and\ B$ be (Q, L)-fuzzy subsets of $G_1$and G_2 respectively such that $\mu_B(y, q) \leq \mu_B(e_1, q)$ holds for all $y \in G_2$, $q \in Q$.e_1 being the identity element of G_1. If $A \times B$ is a (Q, L)-fuzzy subgroup of $G_1 \times G_2$, then B is a (Q, L)-fuzzy subgroup of G_2.

SECTION 5: OTHER PROPERTIES ON (Q, L) – FUZZY SUBGROUPS

Theorem 5.1: Let A, C be (Q, L)-fuzzy subgroups of G_1 and B,D be (Q, L)-fuzzy subgroups of G_2 respectively such that A, C be (Q, L)-fuzzy conjugate subgroups of G_1 and B,D be (Q, L)-fuzzy conjugate subgroups of G_2.Then $A \times B$ of $G_1 \times G_2$ is conjugate to the (Q, L)-fuzzy conjugate subgroup $C \times D$ of $G_1 \times G_2$.

Proof: Since A and C are (Q, L)-fuzzy conjugate subgroups of G_1, $\exists g_1 \in G_1$ such that
$$\mu_A(x, q) = \mu_C({g_1}^{-1}xg_1, q), \forall x \in G_1 .$$

Since B and D are (Q, L)-fuzzy conjugate subgroups of G_2, $\exists g_2 \in G_2$ such that $\mu_B(y, q) = \mu_D({g_2}^{-1}yg_2, q), \forall y \in G_2$.

Now $\mu_{A\times B}\big((x, y), q\big) = \mu_A(x, q) \wedge \mu_B(y, q) = \mu_C({g_1}^{-1}xg_1, q) \wedge \mu_D({g_2}^{-1}yg_2, q)$
$$= \mu_{C\times D}\left(({g_1}^{-1}xg_1, {g_2}^{-1}yg_2), q\right) = \mu_{C\times D}\left(\big({g_1}^{-1}, {g_2}^{-1}\big)(x, y)\big(g_{1,}g_2\big), q\right)$$

Hence the (Q, L) –fuzzy subgroup A X B is conjugate to the (Q, L) –fuzzy subgroup C × D.

Theorem 5.2: Let A be a(Q, L)-fuzzy subset of a group G and V be the strongest fuzzy (Q, L)-fuzzy relation on G. Then A is a(Q, L)-fuzzy subgroup of G iff V is (Q, L)-fuzzy subgroup of $G \times G$.

Proof: Let A be a (Q, L)-fuzzy subgroup of G.

Let $x = (x_1, x_2), y = (y_1, y_2) \in G \times G$. We have

$$\begin{aligned}\mu_V(xy, q) &= \mu_V\big((x_1, x_2)(y_1, y_2), q\big) = \mu_V\big((x_1y_1, x_2y_2), q\big) = \mu_A(x_1y_1, q) \wedge \mu_A(x_2y_2, q)\\ &\geq (\mu_A(x_1, q) \wedge \mu_A(y_1, q)) \wedge (\mu_A(x_2, q) \wedge \mu_A(y_2, q))\\ &= (\mu_A(x_1, q) \wedge (\mu_A(x_2, q)) \wedge (\mu_A(y_1, q) \wedge \mu_A(y_2, q))\\ &= \mu_V\big((x_1, x_2), q\big) \wedge \mu_V\big((y_1, y_2), q\big).\end{aligned}$$

$$\mu_V(xy, q) \geq \mu_V(x, q) \wedge \mu_V(y, q)$$

$$\begin{aligned}\mu_V(x^{-1}, q) &= \mu_V((x_1, x_2)^{-1}, q) = \mu_V\big((x_1^{-1}, x_2^{-1}), q\big)\\ &= \mu_A(x_1^{-1}, q) \wedge \mu_A(x_2^{-1}, q)\\ &= \mu_A(x_1, q) \wedge (\mu_A(x_2, q) = \mu_V(x, q).\end{aligned}$$

Hence V is a (Q, L)-fuzzy subgroup of $G \times G$.

Lemma 5.3: For $a, b \in L$, m is positive integer (i) If a< b, then $a^m < b^m$ (ii) $(a \wedge b)^m = a^m \wedge b^m$

Proof: It is obvious.

Theorem 5.4: Let A be a (Q, L)-fuzzy subgroup of G. Then $A^m = \{< (x, q), (\mu_A(x, q))^m >: x \in G, q \in Q\}$is a (Q, L)-fuzzy subgroup of G^m, where m is a positive integer.

Proof: Let G be a group. Then (G, .) is a group. Hence $(G^m\ ,\ .)$ is also a group

Let A be a (Q, L)-fuzzy subgroup of G. Let $x, y \in G\ and\ q \in Q$
$$\begin{aligned}\mu_{A^m}(xy, q) &= (\mu_A(xy), q)^m \geq (\mu_A(x, q) \wedge \mu_A(y, q))^m\\ &= (\mu_A(x, q)^m \wedge \mu_A(y, q)^m)\\ &= \mu_{A^m}(x, q) \wedge \mu_{A^m}(y, q)\end{aligned}$$

$$\begin{aligned}\mu_{A^m}(x^{-1}, q) &= \big(\mu_A(x^{-1}, q)\big)^m\\ &= (\mu_A(x, q))^m\\ &= \mu_{A^m}(x, q)\end{aligned}$$

Hence A^m is a (Q, L)-fuzzy subgroup of G^m.

Theorem 5.5: If A and $A^C$are (Q, L)-fuzzy subgroups of G, A is a constant (Q, L)-fuzzy subset of G.

Proof: Since A and $A^C$are (Q, L)-fuzzy subgroups of G, it follows that
$$\mu_A(xx^{-1}, q) \geq \mu_A(x, q) \wedge \mu_A(x^{-1}, q)\ , \forall x, x^{-1} \in G, q \in Q \quad (1)$$
$$\mu_{A^C}(xx^{-1}, q) \geq \mu_{A^C}(x, q) \wedge \mu_{A^C}(x^{-1}, q)\ , \forall x, x^{-1} \in G, q \in Q$$

$$\Rightarrow 1-\mu_A(xx^{-1},q) \geq (1-\mu_A(x,q)) \wedge (1-\mu_A(x^{-1},q))$$

$$\Rightarrow 1-[(1-\mu_A(x,q)) \wedge (1-\mu_A(x^{-1},q)] \geq \mu_A(xx^{-1},q)$$

$$\Rightarrow \mu_A(x,q) \vee \mu_A(x^{-1},q) \geq \mu_A(xx^{-1},q) \qquad (2).$$

From (1) & (2) it follows that

$$\mu_A(x,q) \wedge \mu_A(x^{-1},q) \leq \mu_A(xx^{-1},q) \leq \mu_A(x,q) \vee \mu_A(x^{-1},q)$$

$$\Rightarrow \mu_A(x,q) \leq \mu_A(e,q) \leq \mu_A(x,q).$$

$\Rightarrow \mu_A(x,q) = \mu_A(e,q), \forall x \in G, q \in Q$. Therefore A is constant.

Theorem 5.6: IfA^n and A^m are (Q, L)-fuzzy subgroups of G^m, then $A^n \vee A^m$ is also a(Q, L)–fuzzy subgroup of G^m if $n < m$.

Proof: Since $n < m$, then it follows that $A^n \subset A^m$ and $\mu_{A^n}(x,q) \leq \mu_{A^m}(x,q)$.
Now

$$,\mu_{A^n \vee A^m}(xy,q) = \mu_{A^n}(xy,q) \vee \mu_{A^m}(xy,q) = \big(\mu_A(xy,q)\big)^n \vee \big(\mu_A(xy,q)\big)^m$$
$$= (\mu_A(xy,q))^m$$
$$\geq (\mu_A(x,q))^m \wedge (\mu_A(y,q))^m$$
$$= ((\mu_A(x,q))^n \vee (\mu_A(x,q))^m) \wedge ((\mu_A(y,q))^n \vee (\mu_A(x,q))^m)$$
$$= \mu_{A^n \vee A^m}(x,q) \wedge \mu_{A^n \vee A^m}(y,q)$$

$$\mu_{A^n \vee A^m}(x,q) = \mu_{A^n}(x,q) \vee \mu_{A^m}(x,q) = \big(\mu_A(x,q)\big)^n \vee \big(\mu_A(x,q)\big)^m$$
$$= \big(\mu_A(x^{-1},q)\big)^n \vee \big(\mu_A(x^{-1},q)\big)^m$$
$$= \mu_{A^n}(x^{-1},q) \vee \mu_{A^m}(x^{-1},q)$$
$$= \mu_{A^n \vee A^m}(x^{-1},q)$$

Therefore $A^n \vee A^m$ is also a(Q, L) –fuzzy subgroup of G^m .

Theorem 5.7: If $A^n(n = 1,2,\dots.), A^i \subseteq A^j \, for \; i \leq j$ is a (Q, L)–fuzzy subgroup, then $A = A \vee A^2 \vee A^3 \vee \dots$ is also a (Q, L)-fuzzy subgroup.

Proof: Since $A^i \vee A^j$ is also a(Q,L)-fuzzy subgroup for $i \leq j$,also$A^i \subseteq A^j \, for \; i \leq j$.

$Hence, A = A \vee A^2 \vee A^3 \vee$ ...is a (Q,L) –fuzzy subgroup.

CONCLUSION

In this paper, we have discussed the direct product of (Q, L)–fuzzy groups, (Q, L)-fuzzy conjugate groups, and direct product of (Q, L) –fuzzy conjugate groups. Also we have conclude that positive integral powers of a (Q, L)–fuzzy group is a (Q, L)–fuzzy group. This concept can be extended for new results.

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