International Journal of Mathematical Archive-7(11), 2016, 84-88
Available online through www.ijma.info ISSN 2229 – 5046

FIXED POINT THEOREMS FOR $\beta-\psi-\varphi-$CONTRACTIVE TYPE MAPPINGS IN DISLOCATED METRIC SPACES

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(Received On: 20-09-16; Revised & Accepted On: 26-11-16)

ABSTRACT

In this paper we introduced the concept of $\beta-\psi-\varphi-$contractive type mappings in the setting of dislocated metric space. We also provide some example to illustrate our result.

1. INTRODUCTION

Fixed point theory is one of the most dynamic research subject in non linear analysis and many fruitful results have come into the literature in the last few decades. The most remarkable result was given by Banach [1] in 1922 as Banach contraction principle. Later on, many generalization of Banach contraction Principle came into existence in the literature [2-4]. Samet *et al.* [5] introduced the notion of α-ψ contraction mappings and proved the related fixed point theorems.

In 2000, P. Hitzler and A.K. Seda [6] introduced the concept of dislocated metric space and generalized of well known Banach Contraction Principle in this space, which played a key role in the development of logic programming semantics. In this paper, we generalize the concept of α-ψ mappings as $\beta-\psi-\varphi-$contractive mappings in the setting of dislocated metric spaces

2. PRELIMINARIES

Definition 2.1 [5]: Let ψ be a family of functions $\psi: [0,\infty) \to [0,\infty)$ satisfying the following conditions:
(i) ψ is nondecreasing;
(ii) $\sum_{n=1}^{\infty} \psi^n < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ;

Definition 2.2 [5]: Let $T: X \to X$ and $\alpha: X \times X \to [0,\infty)$. we say that T is α-admissible if for all$x, y \epsilon X$, we have
$$\alpha(x,y) \geq 1 \quad \Rightarrow \quad \alpha(Tx,Ty) \geq 1 .$$

Hitzler and Seda [6] introduced the concept of dislocated metric space (d-metric space) as follows:

Definition 2.3 [6]: Let X be a non empty set and let $d: X \times X \to [0,\infty)$ be a function and for all $x, y, z \in X$, the following conditions are satisfied:
(1) $d(x,y) = d(y,x)$;
(2) $d(x,y) = d(y,x) = 0$;
(3) $d(x,y) \leq d(x,z) + d(z,y)$;

Then d is called dislocated metric (or simply d-metric) on X and the pair (X,d) is called dislocated metric space.

Example 2.4 [6]: Let (X,d) be a metric space. The function $f: X \times X \to \mathbb{R}^+$, defined as $d(x,y) = \max(x,y)$; for all $x, y \in X$ is a d-metric on X.

Definition 2.5: A sequence $\{x_n\}$ in a d-metric space (X,d) is said to be d-convergent if for every given $\in > 0$ there exist an $n \in N$ and $x \in X$ such that $d(x_n, x) < \in$ for all $n > N$ and it is denoted by $\lim_{n\to\infty} x_n = x \; or \; x_n \to x \; as \; n \to \infty$

Definition 2.6 [6]: A sequence $\{x_n\}$ in a d-metric space (X,d) is said to be d-Cauchy sequence if for every $\in > 0$ there exist $n_0 \in N$ such that $d(x_n, x_m) < \in$ for all $m, n \in N$.

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Definition 2.7 [6]: A d-metric space (X,d) is called complete if every Cauchy sequence is convergent.

Lemma 2.8 [6]: Let (X,d) be a d-metric space, (x_n) be a sequence in X and $x\in X$. Then $x_n\to x(n\to\infty)$ if and only if $d(x_n,x)\to 0(n\to\infty)$

Lemma 2.9 [6]: Let (X,d) be a d-metric space and let (x_n) be a sequence in X. If the sequence (x_n) is convergent then the limit point is unique.

Theorem 2.10 [6]: Let (X,d) be a complete d-metric space and let $T:X\to X$ be a mapping satisfying the following condition for all $x,y\in X$:

$$d(Tx,Ty)\le kd(x,y),$$

where $k\in[0,1)$. then T has a unique fixed point.

3. MAIN RESULTS

Definition 3.1: Let (X,d) be a d-metric space and $T:X\to X$ be a mapping. T is a $\beta-\psi-\varphi$ contractive type mapping if there exist three functions $\beta:X\times X\to[0,\infty)$ and $\psi,\varphi\in\Psi$ such that

$$\beta(x,y)\psi\big(d(Tx,Ty)\big)\le\psi\big(M(x,y)\big)-\varphi\big(M(x,y)\big) \tag{1}$$

where

$$M(x,y)=max\left\{d(x,y),d(x,Tx),d(y,Ty),\tfrac{d(x,Ty)+d(y,Tx)}{2}\right\};\ \text{for all } x,y\in X. \tag{2}$$

Theorem 3.2: Let (X,d) be a d-metric space and let $T:X\to X$ is a d-β-ψ-contractive mapping of type A and satisfies the following conditions.

(i) T is a β-admissible
(ii) there exist $x_0\in X$ such that $\beta(x_0,Tx_0)\ge 1$;
(iii) T is d-continuous and if $d(x,x)=0$ for some $x\in X$, then $\beta(u,u)\ge 1$.

Then such u is a fixed point of T that is $Tu=u$.

Proof: Let $x_0\in X$ such that $\beta(x_0,Tx_0)\ge 1$ (such a point exist from the condition (ii)). Define the sequence $\{x_n\}$ in X by $x_{n+1}=Tx_n=T^{n+1}x_0$ for all $n\ge 0$. If $x_{n0}=x_{no+1}$ for some n_0, then $u=x_{n0}$ is a fixed point of T. So, we can assume that $x_n\ne x_{n+1}$ for all n. Since T is β-admissible, we have

$$\beta(x_0,x_1)=\beta(x_0,Tx_0)\ge 1 \quad\Longrightarrow\quad \beta(Tx_0,Tx_1)=\beta(x_1,x_2)\ge 1.$$

Inductively, we have

$$\beta(x_n,x_{n+1})\ge 1 \quad \text{for all } n=0,1,2\ldots\ldots \tag{3}$$

From (1) and (3), it follows that for all $n\ge 1$, we have

$$\begin{aligned}\psi\big(d(x_n,x_{n+1})\big)&=\psi(d(Tx_n,Tx_{n+1}))\\&\le\beta(x_n,Tx_{n+1})\psi(d(Tx_n,Tx_{n+1}))\\&\le\psi\big(M(x_n,x_{n+1})\big)-\varphi\big(M(x_n,x_{n+1})\big)\end{aligned} \tag{4}$$

where

$$\begin{aligned}M(x_n,x_{n+1})&=max\left\{d(x_n,x_{n+1}),d(x_n,Tx_n),d(x_{n+1},Tx_{n+1}),\frac{d(x_n,Tx_{n+1})+d(x_{n+1},Tx_n)}{2}\right\}\\&=max\left\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),\tfrac{d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})}{2}\right\};\\&\le max\left\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),\tfrac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+1},x_{n+1})}{2}\right\};\end{aligned}$$

Since $d(x,x)\le d(x,y)$ for each $x,y\in X$, so we have

$$\begin{aligned}M(x_n,x_{n+1})&=max\left\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),\tfrac{2d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{2}\right\};\\&=max\left\{d(x_{n+1},x_{n+2}),\tfrac{2d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{2}\right\};\end{aligned} \tag{5}$$

If for some n, $M(x_n,x_{n+1})=d(x_{n+1},x_{n+2})(=0)$ then from (4) and (5) we have

$$\psi\big(d(x_{n+1},x_{n+2})\big)\le\psi\big(d(x_{n+1},x_{n+2})\big)-\varphi\big(d(x_{n+1},x_{n+2})\big)<\psi\big(d(x_{n+1},x_{n+2})\big),$$

which is not possible. Hence $M(x_n,x_{n+1})=d(x_n,x_{n+1})$ for all $n\in\mathbb{N}$ and (4) and (5) we obtain

$$\psi\big(d(x_{n+1},x_{n+2})\big)\le\psi\big(d(x_n,x_{n+1})\big)-\varphi\big(d(x_n,x_{n+1})\big). \tag{6}$$

Consequently, the sequence $\{d(x_{n+1},x_{n+2})\}$ is non-decreasing for all $n\in\mathbb{N}$.

Taking $n\to\infty$ in (6), and ψ and φ are continuous functions. So we have

$$\lim_{n\to\infty} d(x_{n+1},x_{n+2})=0 \tag{7}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. If it is not, then there exist $\varepsilon>0$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of sequence $\{x_n\}$ where n_k is the smallest index for which

$$n_k>m_k>k \text{ with } d(x_{m_k},x_{n_k})\geq\varepsilon. \tag{8}$$

Then

$$d(x_{m_k},x_{n_k-1})\geq\varepsilon. \tag{9}$$

Using (8) and (9) we obtain

$$\varepsilon\leq d(x_{m_k},x_{n_k})\leq\left[d(x_{m_k},x_{n_k-1})+d(x_{n_k-1},x_{n_k})\right]<\varepsilon+d(x_{n_k-1},x_{n_k}). \tag{10}$$

Taking the upper and lower limit as $k\to\infty$, we conclude

$$\varepsilon\leq\lim_{k\to\infty} inf\, d(x_{m_k},x_{n_k})\leq\lim_{k\to\infty} sup\, d(x_{m_k},x_{n_k})\leq\varepsilon. \tag{11}$$

By the (iii) property of d-metric space, we have

$$d(x_{m_{k+1}},x_{n_k})\leq d(x_{m_{k+1}},x_{m_k})+d(x_{m_k},x_{n_k-1})+d(x_{n_{k-1}},x_{n_k}), \tag{12}$$

with taking the upper limit as $k\to\infty$ in (12), we obtain

$$\lim_{k\to\infty} sup\, d(x_{m_{k+1}},x_{n_k})\leq\varepsilon. \tag{13}$$

By the (iii) property of d-metric space, we have

$$d(x_{m_{k+1}},x_{n_k-1})\leq d(x_{m_{k+1}},x_{m_k})+d(x_{m_k},x_{n_k-1}), \tag{14}$$

By taking the upper limit as $k\to\infty$ in (14),we obtain

$$\lim_{k\to\infty} sup\, d(x_{m_{k+1}},x_{n_k-1})\leq\varepsilon. \tag{15}$$

On the other hand,

$$d(x_{m_k},x_{n_k})\leq d(x_{m_k},x_{m_k+1})+d(x_{m_k+1},x_{n_k-1})+d(x_{n_{k-1}},x_{n_k}). \tag{16}$$

Using (11) and (7), we obtain

$$\varepsilon\leq\lim_{k\to\infty} inf\, d(x_{m_k+1},x_{n_k-1}). \tag{17}$$

Moreover,

$$\varepsilon\leq\ d(x_{m_k},x_{n_k})\leq\ d(x_{m_k},x_{m_k+1})+d(x_{m_k},x_{n_k}), \tag{18}$$

with taking the upper limit as $k\to\infty$ in (18), we have

$$\varepsilon\leq\lim_{k\to\infty} sup d(x_{m_k},x_{n_k}). \tag{19}$$

By using (1), we have

$$\begin{aligned}\psi\left(d(x_{m_k+1},x_{n_k})\right)&\leq\beta(x_{m_k},x_{n_k-1})\psi\left(d(Tx_{m_k},Tx_{n_k-1})\right),\\&\leq\psi\left(M(x_{m_k},x_{n_{k-1}})\right)-\varphi\left(M(x_{m_k},x_{n_{k-1}})\right).\end{aligned} \tag{20}$$

where

$$M(x_{m_k},x_{n_{k-1}})=max\left\{d(x_{m_k},x_{n_{k-1}}),d(x_{m_k},x_{m_{k+1}}),d(x_{n_{k-1}},x_{n_k}),\frac{d(x_{m_k},x_{n_k})+d(x_{n_{k-1}},x_{m_{k+1}})}{2}\right\} \tag{21}$$

From on taking the upper limit as $k\to\infty$, from (7), (9), (11) and (15) we obtain

$$\lim_{k\to\infty} sup M(x_{m_k},x_{n_{k-1}})=max\{\varepsilon,0,0,\varepsilon\}=\varepsilon. \tag{22}$$

Thus, from (19) and (20), we have

$$\psi(\varepsilon)\leq\psi(\varepsilon)-\varphi(\varepsilon) \tag{23}$$

which is not possible. Hence $\{x_n\}$ is a cauchy sequence in X. Since X is complete, there exists $z\varepsilon X$ such that

$$0=\lim_{n,m\to\infty} d(x_n,x_m)=\lim_{n\to\infty} d(x_n,z)=d(z,z). \tag{24}$$

By using (iii) property of d-metric space, we have

$$d(z,Tz)\leq d(z,Tx_n)+d(Tx_n,Tz). \tag{25}$$

Taking the upper limit as $n \to \infty$ in (25) and using the continuity of T we have
$$d(z,Tz) \le d(Tz,Tz) \tag{26}$$

Since $d(z,z) \ge 1$ and using (1) we have
$$\psi\big(d(Tz,Tz)\big) \le \beta(z,z)\psi\big(d(Tz,Tz)\big) \le \psi\big(M(z,z)\big) - \varphi\big(M(z,z)\big) \tag{27}$$
where
$$M\big(x_{m_k}, x_{n_{k-1}}\big) = max\left\{d(z,z), d(z,Tz), d(z,Tz), \frac{d(z,Tz)+d(z,Tz)}{2}\right\} = d(z,Tz). \tag{28}$$

Hence
$$\psi\big(d(Tz,Tz)\big) \le \beta(z,z)\psi\big(d(Tz,Tz)\big) \le \psi\big(d(z,z)\big) - \varphi\big(d(z,z)\big). \tag{29}$$

The property of ψ, we obtain
$$d(Tz,Tz) \le d(z,Tz). \tag{30}$$

Hence we deduce $\varphi\big(d(z,Tz)\big) = 0$. Hold
$$d(Tz,z) = d(Tz,Tz) = d(Tz,z) = 0 \text{ and } Tz = z.$$

Hence, z is a fixed point of T.

If we replace the continuity condition (iii), Theorem 1 remains true. This statement is given as follows.

Theorem 3.3: Let (X,d) be a d-metric space and let $T: X \to X$ is a d-β-ψ-contractive mapping of type A and satisfies the following conditions.

(i) T is a β-admissible
(ii) there exist $x_0 \in X$ such that $\beta(x_0, Tx_0) \ge 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\beta\big(x_{n_k}, x\big) \ge 1$ for all k.

Then, such z is a fixed point of T, that is $Tz = z$.

Proof: From proof of Theorem 3.3, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ is cauchy in (X,d) and converges to some $z \in X$. Consider (24),
$$\lim_{k\to\infty} d\big(x_{n_{k+1}}, Tz\big) = d(z,Tz) \tag{31}$$
holds. By the assumption on X, we have
$$\begin{aligned}\psi\left(d\big(x_{n_{k+1}}, Tz\big)\right) &\le \beta\big(x_{n_k}, z\big)\psi\left(d\big(Tx_{n_k}, Tz\big)\right)\\ &\le \psi\left(M\big(x_{n_k}, z\big)\right) - \varphi\left(M\big(x_{n_k}, z\big)\right)\end{aligned} \tag{32}$$
where
$$\begin{aligned}M\big(x_{n_k}, z\big) &= max\left\{d\big(x_{n_k}, z\big), d\big(x_{n_k}, Tx_{n_k}\big), d(z,Tz), \frac{d\big(x_{n_k},Tz\big)+d\big(z,Tx_{n_k}\big)}{2}\right\},\\ &= max\left\{d\big(x_{n_k}, z\big), d\big(x_{n_k}, x_{n_{k+1}}\big), d(z,Tz), \frac{d\big(x_{n_k},Tz\big)+d\big(z,x_{n_{k+1}}\big)}{2}\right\}.\end{aligned}$$
with (7) and (31), we have
$$\lim_{k\to\infty} M\big(x_{n_k}, z\big) = d(z,Tz). \tag{33}$$

Since $\beta(x_n, z) \ge 1$ we have
$$\begin{aligned}\psi d(Tz,z) &\le \psi\big(d(Tz,Tx_n) + d(Tx_n,z)\big)\\ &\le \psi\big(d(Tz,Tx_n)\big) + \psi\big(d(Tx_n,z)\big)\\ &\le \beta(z,x_n)\psi\big(d(Tz,Tx_n)\big) + \psi\big(d(Tx_n,z)\big)\\ &\le \psi\big(M(z,x_n)\big) - \varphi\big(M(z,x_n)\big)\end{aligned} \tag{34}$$

Let $n \to \infty$ in (34), we have $\psi(d(Tz.z) \le 0)$. hence z is a fixed point of T, or equivalently, $z = Tz$.

Corollary 3.4: Let (X,d) be a d-metric space and $T: X \to X$ be such that
$$d(Tx,Ty) \le M(x,y) - \varphi\big(M(x,y)\big)$$
for all $x, y \in X$ where $M(x,y)$ defined by (2). Then, T has a fixed point.

Proof To prove this corollary it is suffices to take $\beta(x,y) = 1$ and $\psi(t) = t$ in theorem 3.4.

Example 3.5: Let $X=[0,\infty)$ be the d-metric space, where $d(x,y)=|x-y|$ for all $x,y\in X$. Consider the self-mapping $T:X\to X$ given by $Tx=\frac{\ln(x+1)}{2}$.

Define $\beta: X\times X\to[0,\infty)$, as

$$\beta(x,y)=\begin{cases}1 & if\ x,y\in[0,1],\\ 0 & otherwise.\end{cases}$$

Define the mappings $\psi,\varphi:[0,\infty)\to[0,\infty)$ by $\psi(t)=t,\varphi(t)=\frac{t}{2}$.

It is easy to see that T is a continuous on X and also T is a $\beta-\psi-\varphi-$contractive type mapping with $\psi(t)=t,\varphi(t)=\frac{t}{2}$ for all $t\geq 0$, for $x,y\in X$,

$$\begin{aligned}\beta(x,y)\psi\big(d(Tx,Ty)\big)&=\psi\big(d(Tx,Ty)\big)\\&=d(Tx,Ty)=|Tx,Ty|\\&=\left|\frac{\ln(x+1)}{2}-\frac{\ln(y+1)}{2}\right|\\&\leq\left|\frac{x}{2}-\frac{y}{2}\right|=\frac{1}{2}d(x,y)=d(x,y)-\frac{1}{2}d(x,y)\\&=\psi\big(d(x,y)\big)-\varphi\big(d(x,y)\big)\\&\leq\psi\big(M(x,y)\big)-\varphi\big(M(x,y)\big).\end{aligned}$$

Now, we claim that T is β −admissibe.

Let $(x,y)\in X\times X$ such that $\beta(x,y)\geq 1$. From the definition of T and β we have both $Tx=\frac{\ln(x+1)}{2}$ and $Ty=\frac{\ln(y+1)}{2}$ are in $[0,1]$. Therefore $\beta(Tx,Ty)=1$. Then $T$is β −admissible.

So (i) hypothesis of the theorem is satisfied.

Now taking $x_0=0$ and $Tx_0=T0=\frac{\ln(0+1)}{2}=0$, we have $\beta(x_0Tx_0)=\beta(0,T0)=1\geq 1$.

So (ii) hypothesis of theorem is also satisfied and obviously (iii) hypothesis is also satisfied.

So all the condition of theorem is satisfied and has a fixed point $z=0$.

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Source of support: Nil, Conflict of interest: None Declared