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# EXISTENCE OF FIXED POINTS

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***ABSTRACT***

*In this paper we consider a class of self-mappings on Cone Banach Space which have at least one fixed point. That is, for a convex and closed subset C of a Cone Banach Space with the norm* $\|\ x\ \|_p = d(x, 0)$, *if there exists elements* $a, b, s$ *and a mapping* $T: C \to C$ *satisfying the conditions*

$$0 \le s + |\, a\, | - 2b < 2(a + b) \text{ and } ad(Tx, Ty) + b\{d(x, Tx) + d(y, Ty)\} \le s\, d(x, y) \text{ for all } x, y \in C$$

*then* $T$ *has at least one fixed point.*

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## 1. INTRODUCTION AND STATEMENT OF RESULTS

Rzepecki [12] introduced a generalized metric $d_E$ on a set $X$ in such a way that $d_E: X \times X \to S$, where E is a Banach Space and S is a normal on E with partial order $\le$. In that paper, the author generalized the fixed point theorems of Maia type [10]. i.e. let $X$ be a non empty set endowed in two metrics $d_1, d_2$ and $T$ a mapping of $X$ into itself. Suppose that $d_1(x, y) \le d_2(x, y)$ for all $x, y \in X$, and $X$ is complete space with respect to $d_1$ and $T$ is continuous w.r.t. $d_1$ and $T$ is contraction w. r. t. $d_2$, that is, $d_2(Tx, Ty) \le k\, d_2(x, y)$ for all $x, y \in X$, where $0 \le k < 1$. Then $f$ has a unique fixed point in $X$.

After a long period of eight years, Lin [13] considered the notion of K-metric spaces by replacing real numbers with cone K in the metric function, that is, $d: X \times X \to K$. In his paper, some results of Khan and Imdad [11] on fixed point theorems were considered for K-metric spaces. Without mentioning the papers of Lin and Rzepecki, in 2007, Huang and Zhang [8] announced the notion of cone metric spaces (CMS) by replacing real numbers with an ordering Banach Space. In that paper, they also discussed some properties of convergence sequences and proved the fixed point theorems of contractive mapping for cone metric spaces: *i.e.* any mapping $T$ of a complete cone metric space $X$ into itself that satisfies, for some $0 \le k < 1$, the inequality

$$d(Tx, Ty) \le k\, d(x, y) \tag{1.1}$$

for all $x, y \in X$, has a unique fixed point.

Recently, many results on fixed point theorems have been extended to cone metric spaces [8, 9].

In this paper, some of known results [7, 15] are extended to Cone Banach Spaces which were defined and used in [3, 16] where the existence of fixed point for self-mapping on Cone Banach Spaces is investigated.

In this paper, $E = (E, \|.\|)$ stands for real Banach Space and $P = P_E$ always be a closed non empty subset of $E$.

Here $P$ is called cone if $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers $a, b$ where $P \cap (-P) = \{0\}$ and $P \ne \{0\}$.

Now for a given cone P, we can define a partial ordering denoted by $\le or \le p$ with respect to P by $x \le y\ iff\ y - x \in P$. The notation $x < y$ indicates that $x \le y$ will show $y - x \in int.P$; where $int.P$ denotes the interior of P. From now it is assumed that $int.P \ne \phi$.

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The cone P is called (N) normal if there is a number $K \geq 1$ s.t. $\forall\, x, y \in E,\ 0 \leq x \leq y$ implies that

$$\| x \| \leq K \| y \| \tag{1.2}$$

And (R) regular, if every increasing sequence which is bounded above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq x_3 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that. $lim_{n\to\infty} \| x_n - x \| = 0$.

In above (N), the least positive integer K, satisfying (1.2), is called the normal constant of P.

Now we have the following lemma.

**Lemma 1.1 [14]:**
(a) Every regular cone is normal.
(b) For each k>1, there is a normal cone with normal constant K > k.
(c) The cone P is regular if every decreasing sequence which is bounded below is convergent.
Here the proofs of (a), (b) are given in [14] and that of (c) follows from definition.

Now we have the following definitions.

**Definition 1.2 [8]:** Suppose X is a non-empty set and the mapping $d: X \times X \to E$ satisfies the following four onditions:
(a) $0 \leq d(x, y)$ for all $x, y \in X$
(b) $d(x, y) = 0 \ \ iff\ x = y$
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$
(d) $d(x, y) = d(y, x)$ for all $x, y \in X$

Then $d$ is called cone metric on $X$ and the pair $(X, d)$ is called a cone metric space (CMS).

Now we define cone normal spaces.

**Definition 1.3 [3, 16]:** Consider $X$ to be a vector space over R and let the mapping $\|.\|_p: X \to E$ satisfies the following four conditions:
(a) $\| x \|_p > 0 \ \ \forall \ \ x \in X$
(b) $\| x \|_p = 0 \ \ iff \ \ x = 0$
(c) $\| x + y \|_p \leq \| x \|_p + \| y \|_p$ for all $x, y \in X$
(d) $\| kx \|_p = |k| \| x \|_p \ \ \forall \ \ k \in R$

Then $\|.\|_p$ is called cone norm on $X$, and the pair $(X, \|.\|_p)$ is called a cone normed space (CNS). Here we observe that every CNS is CMS, in fact $d(x, y) = \| x - y \|_p$.

We again have an important definition, which we use in this paper.

**Definition 1.4:** Suppose $(X, \|.\|_p)$ be a CNS, $x \in X$ and $\{x_n\}$ be a sequence in $X$, then
(i) $\{x_n\}$ Converges to $x$, whenever for every $c \in E$ with $0 \ll c$, there is a natural number N, such that $\| x_n - x \|_p \ll c \ \forall\, n \geq N$. It is also denoted by $lim_{n\to\infty} x_n = x$ or $x_n \to x$
(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number N, such that $\| x_n - x \|_p \ll c$ for all $n, m \geq N$.
(iii) $(X, \|.\|_p)$ is a complete cone normed space if every Cauchy sequence is convergent. Here complete cone normed spaces will be called cone Banach spaces.
Now we have the following lemmas and definitions.

**Lemma 1.5:** If $(X, \|.\|_p)$ is a CNS, $\{x_n\}$ a sequence in X and P, a normal cone with normed constant K, then
(i) the sequence $\{x_n\}$ converges to $x \ \ iff \ \| x_n - x \|_p \to 0$ as $n \to \infty$.
(ii) the sequence $\{x_n\}$ is Cauchy $iff \ \| x_n - x_m \|_p \to 0$ as $n, m \to \infty$ and
(iii) If the sequence $\{x_n\}$ converges to $x$ and the sequence $\{y_n\}$ converges to $y$, then $\| x_n - y_n \|_p \to \| x - y \|_p$.

Proofs of the above are got by applying [8, Lemmas 11, 12 & 8] to the cone metric space $(X, d)$ where $d(x, y) = \| x - y \|_p \ \ \forall \ \ x, y \in X$.

**Lemma 1.6 [1, 2]:** Suppose $(X, \|.\|_p)$ be a CNS over a cone P in E. Then we have the following four results:
(a) $int.(P) + int.(P) \subseteq int.(P)$ and $\lambda.int.(P) \subseteq int.(P)$; $\lambda > 0$
(b) If $c \gg 0$, then there exists a $\delta > 0$ such that $\| b \| < \delta$ implies that $b \ll c$.
(c) For any given $c \gg 0$ and $c_0 \gg 0$, $there\ exists\ \ n_0 \in N$ such that ${c_0}/{n_0} \ll c$.
(d) If $\{a_n\}, \{b_n\}$ are two sequences in E such that $a_n \to a,\ \ b_n \to b$ and $a_n \leq b_n$ for all $n$ then $a \leq b$.

 

**Definition 1.7:** P is called minihedral cone if $\sup\{x, y\}$ exists for all $x, y \in E$, and strongly minihedral if every subset of E which is bounded above has a supremum.

**Lemma 1.8 [2]:** Every strongly minihedral normal cone is regular.

**Definition 1.9:** Suppose C be a closed and convex subset of a Cone Banach Space with the norm $\| x \|_p = d(x, 0)$ and $T: C \to C$ be a mapping which satisfies the condition

$\frac{1}{2} \| x - Tx \|_p \leq \| x - y \|_p$. This implies that implies

$$\| Tx - Ty \|_p \leq \| x - y \|_p \tag{1.3}$$

for all $x, y \in C$. Then T is said to satisfy the condition (C). For $T: X \to X$, the set of fixed points of T is denoted by

$$F(T) = \{z \in X: Tz = z\}.$$

**Definition 1.10 [15]:** Suppose C be a closed and convex subset of a Cone Banach Space with the norm $\| x \|_p = d(x, 0)$ and $T: C \to C$ be a mapping. Consider the conditions

$$\| Tx - Tz \|_p \leq \| x - z \|_p \quad \forall\, x, z \in C \tag{1.4}$$

$$\| Tx - z \|_p \leq \| x - z \|_p \quad \forall\ x \in C\ ;\ z \in F(T) \tag{1.5}$$

then T is called non-expansive if it satisfies the condition (1.4) .

Now we have the main results and their proofs.

## 2. MAIN RESULTS

Here in main results we represent a Cone Banach Space by $X = (X, \|.\|_p)$, a normal cone with constant K by P and a self-mapping operator defined on a subset C of X by T.

**Theorem 2.1:** Suppose $a \in R$, $a > 1$; $(X, d)$ be a complete cone metric space and $T: X \to X$, an onto mapping which satisfies the condition

$$d(Tx, Ty) \geq a.\, d(x, y) \tag{2.1}$$

then T has a unique fixed point.

**Proof of 2.1:** If $x \neq y$ and $Tx = Ty$, then from (2.1), we see that $0 \geq a.\, d(x, y)$ which is a contradiction. Then T is one-to-one and it has an inverse, say S. Thus we have

$$d(x, y) \geq a.\, d(Sx, Sy) \Leftrightarrow d(Sx, Sy) \leq \frac{1}{a} d(x, y) \tag{2.2}$$

By [8, theorem 1], S has a unique fixed point which in other words means that T has a unique fixed point. This proves theorem 2.1.

Now we have the two propositions:

**Proposition 2.2:** Every non-expansive mapping satisfies the condition (C), this statement is a consequence of definition (1.9).

**Proposition 2.3:** If T satisfy the condition (C) and $F(T) \neq \varphi$, then T is a quasi-non-expansive.

**Proof of 2.3:** Let $z \in F(T) and\ x \in C$. Since $\frac{1}{2} \| z - Tz \|_p = 0 \leq \| z - x \|_p$ and satisfies the condition (C),

$$\| z - Tx \|_p = \| Tz - Tx \|_p \leq \| z - x \|_p \tag{2.3}$$

This proves proposition (2.3).

Now we prove one more theorem.

**Theorem 2.4:** Let C be a closed and convex subset of a Cone Banach Space $X$ with the norm $\| x \|_p = d(x, 0)$ and $T: C \to C$ be a mapping which satisfies the condition

$$d(x, Tx) + d(y, Ty) \leq q.\, d(x, y) \tag{2.4}$$

for all $x, y \in C$, where $2 \leq q < 4$. Then T has at least one fixed point.

**Proof of theorem 2.4:** Let $x_0$ be an arbitrary point in $C$, we define a sequence $\{x_n\}$ as follows;

$$x_{n+1} = \frac{x_n + T(x_n)}{2}, \qquad n = 0,1,2, \ldots \tag{2.5}$$

We note that

$$x_n - Tx_n = 2\{x_n - \left(\frac{x_n + Tx_n}{2}\right)\} = 2(x_n - x_{n+1}) \tag{2.6}$$

which gives

$$d(x_n, Tx_n) = \| x_n - Tx_n \|_p = 2 \| x_n - x_{n+1} \|_p$$
$$= 2\, d(x_n, x_{n+1}) \;\; for\; n = 0,1,2,\ldots \tag{2.7}$$

Combining this observation with the condition (2.4), then we get

$$2d(x_{n-1}, x_n) + 2d(x_n, x_{n+1}) \le q.\, d(x_{n-1}, x_n) \tag{2.8}$$

Hence, $d(x_n, x_{n+1}) \le k.\, d(x_{n-1}, x_n)$ where $k = \frac{q-2}{2} < 1$.

Thus, $\{x_n\}$ is a Cauchy sequence in C and thus converges to some point $z \in C$.

Regarding the inequality

$$d(z, Tx_n) \le d(z, x_n) + d(x_n, Tx_n) = d(z, x_n) + 2d(x_n, x_{n+1}) \tag{2.9}$$

and with the help of lemma 1.5 (iii), we have

$$Tx_n \to z \tag{2.10}$$

Considering (2.6) and (2.4) and putting $x = z$ and $y = x_n$ implies that

$$d(z, Tz) + 2d(x_n, x_{n+1}) \le q.\, d(z, x_n) \tag{2.11}$$

Hence, if $n \to \infty$, we can have $d(z, Tz) \le 0 \Rightarrow Tz = z$. This prove theorem 2.4.

We observe that the identity mapping $I(x) = x$, satisfies the condition (2.4). Thus maps that satisfy the condition (2.4) many have fixed points. We have by triangle inequality that

$$d(x, Tx) + d(y, Ty) \le d(x, y) + d(y, Tx) + d(y, x) + d(x, Ty) \tag{2.12}$$

By (2.4), we have

$$d(x, Tx) + d(y, Ty) \le 2\, d(x, y) + q\, d(x, y) = (2 + q)\, d(x, y),\;\; 2 \le q < 4 \tag{2.13}$$

We put $p = 2 + q$ and obtain

$$d(x, Tx) + d(y, Ty) \le 2d(x, y) + q.\, d(x, y) = p.\, d(x, y), \tag{2.14}$$

Due to this, we have the following new theorem.

**Theorem 2.5:** If C is a closed and convex subset of a Cone Banach Space with the norm $\| x \|_p = d(x, 0)$ and $T: C \to C$ is a mapping which satisfies the condition

$$d(x, Tx) + d(y, Ty) \le p.\, d(x, y) \tag{2.15}$$

for all $x, y \in C$, where $0 \le p < 2$. Then T has a fixed point.

**Theorem 2.6:** If C is a closed and convex subset of a Cone Banach Space with the norm $\| x \|_p = d(x, 0)$ and $T: C \to C$ is a mapping which satisfies the condition

$$d(Tx, Ty) + d(x, Tx) + d(y, Ty) \le r.\, d(x, y) \tag{2.16}$$

for all $x, y \in C$, where $2 \le r < 5$. Then T has at least one fixed point.

**Proof of Theorem 2.6:** We form a sequence $\{x_n\}$ as in the proof of theorem 2.4, i.e. (2.5), (2.6) and also

$$x_n - Tx_{n-1} = \frac{x_{n-1+Tx_{n-1}}}{2} - Tx_{n-1} = \frac{x_{n-1+Tx_{n-1}}}{2}$$
$$d(x_n, Tx_{n-1}) = \| x_n - Tx_{n-1} \|_p = \frac{1}{2} \| x_n - Tx_{n-1} \|_p = \frac{1}{2} d(x_{n-1}, Tx_{n-1}) \tag{2.17}$$

hold.

Thus, by triangle inequality, we have

$$d(x_n, Tx_n) \le d(Tx_{n-1}, Tx_n) + d(x_n, Tx_{n-1}), \text{ that is}$$
$$d(x_n, Tx_n) - d(x_n, Tx_{n-1}) \le d(Tx_{n-1}, Tx_n) \tag{2.18}$$

We have the inequality (by 2.7 & 2.17)

$$2d(x_n, x_{n+1}) - d(x_n, x_{n-1}) \le d(Tx_{n-1}, Tx_n) \tag{2.19}$$

Putting $x = x_{n-1}$ and $y = x_n$ in (2.16), we have (by 2.19 & 2.7) that

$$2d(x_n, x_{n+1}) + d(x_n, x_{n-1}) - 2d(x_n, x_{n-1}) + 2d(x_n, x_{n+1}) \le r.\, d(x_{n-1}, x_n) \tag{2.20}$$

Which further gives that

$$d(x_n, x_{n+1}) \le \{\tfrac{r-1}{4}\}\, d(x_n, x_{n-1}).$$

Since $1 \le r < 5$, the sequence $\{x_n\}$ is a Cauchy sequence that converges to some point $z \in C$. Since $\{Tx_n\}$ also converges to $z$ as in the proof of theorem 2.4, the inequality (2.16) (with the assumption $x = z, y = x_n$) by the help of lemma (1.5) (iii) yields that

$$d(Tz, z) + d(z, Tz) \le 0$$ which implies that $Tz = z$. Hence theorem (2.6) is proved.

Now we prove the last theorem in this paper.

**Theorem 2.7:** Let C be a closed and convex subset of a Cone Banach Space with the norm $\| x \|_p = d(x, 0)$. If there exists elements $a, b, s$ and a mapping $T: C \to C$ which satisfies two conditions

$$0 \le s + |a| - 2b < 2(a + b) \tag{2.21}$$

and

$$a.d(Tx, Ty) + b\{d(x, Tx) + d(y, Ty)\} \le s.d(x, y) \tag{2.22}$$

for all $x, y \in C$. Then T has at least one fixed point.

**Proof of theorem 2.7:** For the proof of this theorem we make a sequence $\{x_n\}$ as in the proof of theorem 2.4 and we claim that the inequality (2.22) for $x = x_{n-1}, y = x_n$ implies that

$$2a.d(x_n, x_{n+1}) - |a|.d(x_{n-1}, x_n) + 2b\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \le s.d(x_{n-1}, x_n) \tag{2.23}$$

for all $a, b, s$ that satisfy (2.21). For the proof of the claim, we have from (2.7) that

$$\left.\begin{aligned} d(x_{n-1}, Tx_{n-1}) &= 2d(x_{n-1}, x_n) \\ d(x_n, Tx_n) &= 2d(x_n, x_{n+1}) \end{aligned}\right\} \tag{2.24}$$

The case when $a \ge 0$ is trivially true. In fact, on considering (2.22) with $x = x_{n-1}$ and $y = x_n$ together with (2.24) and (2.19), we can show

$$2a.d(x_n, x_{n+1}) - a.d(x_{n-1}, x_n) + 2b\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \le s.d(x_{n-1}, x_n) \tag{2.25}$$

which is equivalent to (2.23) , since $|a| = a$.

For the case $a < 0$, we consider the inequality

$d(Tx_{n-1}, Tx_n) \le d(x_n, Tx_n) + d(x_n, Tx_{n-1})$ which is equivalent to

$$a\{d(x_n, Tx_n) + d(x_n, Tx_{n-1})\} \ge a.d(Tx_{n-1}, Tx_n) \tag{2.26}$$

By putting $x = x_{n-1}$ and $y = x_n$ in (2.22) together with (2.24), (2.26) and (2.17), we can show that

$$2a.d(x_n, x_{n+1}) + a.d(x_{n-1}, x_n) + 2b\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \le s.d(x_{n-1}, x_n) \tag{2.27}$$

which is clearly equivalent to (2.23) since $|a| = -a$. Hence our claim is proved.

From (2.23), we can have

$$d(x_n, x_{n+1}) \le \frac{|a| - 2b + s}{2(a + b)} d(x_{n-1}, x_n) \tag{2.28}$$

By (2.21), we have

$$0 \le \frac{|a| - 2b + s}{2(a + b)} < 1.$$

Hence, the sequence $\{x_n\}$ is a Cauchy sequence that converges to some point $z \in C$. By replacing $x$ with $z$ and $y$ with $x_n$ in (2.22), we can prove that

$$a.d(Tz, z) + b.d(z, Tz) \le 0 \quad \text{as } n \to \infty \tag{2.29}$$

From (2.29), we have $Tz = z$ as $(a + b) > 0$ which prove theorem (2.7) completely.

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