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# SINGLE SERVER BULK QUEUE WITH FEEDBACK, TWO CHOICES OF SERVICE AND COMPULSORY VACATION

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***ABSTRACT***

*The concept of this paper deals with the customers arriving in bulk or group, in a single server queueing system, in Poisson distribution which provides two types of general services in bulk of fixed size $\boldsymbol{M}$ $(\geq \mathbf{1})$ in first come first served basis. After first stage service, the server must provide the second stage service. However after the completion of second stage service, if the batch of customer is dissatisfied with his service, he can immediately join the tail of the queue as a feedback customer with probability $\boldsymbol{p}$ to repeat the service until it is successful. Otherwise the batch of customer may depart the system with probability $\mathbf{1}-\boldsymbol{p}$. After completion of the service, the server takes compulsory vacation under general distribution. If the required bulk of customers is not available on the return of the server, the server again goes for vacation or remains in the system till bulk is reached. Taking this concept into account, we derive the time dependent probability generating functions and from it the corresponding steady state results are obtained. The average queue size, the system size and numerical results are discussed.*

***AMS Mathematics subject classification:*** *60K25, 68M20, 90B22.*

***Keywords—****bulk arrival; bulk service; vacation; feedback.*

## I. INTRODUCTION

Among the renowned researchers, Chaudhry and Templeton [4], Cooper [5], Gross and Harris [8] have done researches on bulk queues. Vacation queue has been surveyed broadly by Doshi [6]. The queueing system with compulsory vacation has been explained by Madan [12].Thangaraj and Vanitha[13] have worked the M/G/1 Queue with Two Stage Heterogeneous Service Compulsory Server Vacation. Saravanarajan and Chandrasekaran [17], Thangaraj and Vanitha [18] have discussed $\mathrm{M/G/1}$ feedback queue with two types of services. Bulk service queues with vacation and feedback have described by Kalyanaraman, Lakshmi Srinivasan and Sekar [9]. Ayyappan and Sathiya [1], Maragathasundari and Srinivasan [15] have worked about feedback queue with server vacations

In this paper, we propose to study Single Server Bulk Queue with Feedback, Two Choices of Service and Compulsory Vacation. The arrival is under Poisson distribution, the services and compulsory vacation are in general distribution. The bulk of customers are served under first come first served basis. After first stage service, the server must provide the second stage service. However after the completion of second stage service, if the batch of customer is dissatisfied with his service, he can immediately join the tail of the queue as a feedback customer with probability p to repeat the service until it is successful. Otherwise the batch of customer may depart the system with probability $1-\mathrm{p}$.

In day to day life, one encounters numerous examples of the queueing situations where all arriving batch of customers require the type 1 service and type 2 service continuously. Some of the batch of customers, who is not satisfied with the service, joins the tail of the original queue again, until the service is successful. Otherwise the batches of customers leave the system.

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These phenomena are observed in situations involving many real-time applications like production system, traffic signals, communication system, waiting lines at railway reservation service, banking service and so forth. We consider below some such queueing situations:

1. In signals and systems, the data is sent to the receiver through stage 1 signal and stage 2 signal. If the receiver is unavailable, the data rejoin the tail of the stage 1 till the receiver demand for it.
2. In a hospital, patients are arriving for treatment for particular diseases. All of them are sent to the doctor and they are given medicines. Patients whose disease is possible to be cured will leave the hospital. The rest of the patients are to be sent to the doctor again.

This paper is ordered as follows: The mathematical model is explained in section 2. Definitions and Notations are briefed in section 3. Equations governing the system are given in section 4. The time dependent solutions have been derived in section 5 and corresponding steady state results have been calculated clearly in section 6. The average queue size and the system size are computed in section 7. Numerical results are discussed in section 8.

## II. THE MATHEMATICAL MODEL

We assume the following to describe the queueing model of our study:

(a) Customers (units) arrive at the system in batches of variable size in a compound Poisson process.

(b) Let $\lambda\pi_i dt$ $(i = 1,2,3,\dots)$ be the first order probability that a batch of $i$ customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq \pi_i \leq 1$, $\sum_{i=1}^{\infty} \pi_i = 1$, $\lambda > 0$ is the mean arrival rate of batches.

(c) We consider the case when there is single server providing parallel service of two types on a first come first served basis (FCFS).

(d) The service of customers (units) is rendered in batches of fixed size $M(\geq 1)$ or $\min(n, M)$, where $n$ is the number of customers in the queue.

(e) After first stage service, the server must provide the second stage service. However after the completion of second stage service, if the batch of customer is dissatisfied with his service, he can immediately join the tail of the queue as a feedback customer with probability p to repeat the service until it is successful. Otherwise the batch of customer may depart the system with probability $1 - \text{p}$.

(f) We assume that the random variable of service time $S_j\,(j = 1,2)$ of the $j^{th}$ kind of service follows a general probability law with distribution function $G_j(s_j)$, $g_j(s_j)$ is the probability density function and $E(S_j^k)$ is the $k^{th}$ moment $(k = 1,2,\dots)$ of service time $j = 1,2$.

(g) Let $\mu_j(x)$ be the conditional probability of type $j$ service completion during the period $(x, x + dx]$, given that elapsed service time is $x$, so that

$$\mu_j(x) = \frac{g_j(x)}{1-G_j(x)}, j = 1,2 \tag{1}$$

and therefore

$$g_j(s_j) = \mu_j(s_j) e^{-\int_0^{s_j} \mu_j(x)dx}, j = 1,2 \tag{2}$$

(h) After completion of continuous service to the batches of fixed size $M(\geq 1)$, the server will go for compulsory vacation.

(i) We further assume that the random variable of vacation time $Y$ follows a general probability law with distribution function $V(y)$, $v(y)$ is the probability density function and $E(Y^k)$ is the $k^{th}$ moment $(k = 1,2,\dots)$ of vacation time.

(j) Let $\alpha(x)$ be the conditional probability of completion of a vacation period during the interval $(x, x + dx]$, given that the elapsed vacation time is $x$, so that

$$\alpha(x) = \frac{v(x)}{1-V(x)} \tag{3}$$

and therefore

$$v(y) = \alpha(y) e^{-\int_0^{y} \alpha(x)dx} \tag{4}$$

(k) On returning from vacation the server instantly starts the service if there is a batch of size $M$ or he remains idle in the system.

(l) Finally, it is assumed that the inter-arrival times of the customers, the service times of each kind of service and vacation times of the server, all these stochastic processes involved in the system are independent of each other.

## III. DEFINITIONS AND NOTATIONS

We define:

$P_{n,j}(x,t)$ : Probability that at time $t$, the server is active providing and there are $n$ $(n \geq 0)$ customers in the queue, excluding a batch of $M$ customers in type $j$ service, $j = 1,2$ and the elapsed service time of this customer is $x$.

Accordingly, $P_{n,j}(t) = \int_0^\infty P_{n,j}(x,t)dx$ denotes the probability that there are $n$ customers in the queue excluding a batch of $M$ customers in type $j$ service, $j = 1,2$ irrespective of the elapsed service time $x$.

$V_n(x,t)$ : Probability that at time $t$, there are $n$ $(n \geq 0)$ customers in the queue and the server is on vacation with the elapsed vacation time $x$. Accordingly, $V_n(t) = \int_0^\infty V_n(x,t)dx$ denotes the probability that there are $n$ customers in the queue and the server is on vacation irrespective of the value of $x$.

$Q(t)$ : Probability thatat time $t$, there are less than $M$ customers in the system and the server is idle but available in the system.

## IV. EQUATIONS GOVERNING THE SYSTEM

According to the Mathematical model mentioned above, the system has the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_{n,1}(x,t) + \frac{\partial}{\partial t}P_{n,1}(x,t) + \big(\lambda + \mu_1(x)\big)P_{n,1}(x,t) = \lambda \sum_{k=1}^{n} \pi_k P_{n-k,1}(x,t) \tag{5}$$

$$\frac{\partial}{\partial x}P_{0,1}(x,t) + \frac{\partial}{\partial t}P_{0,1}(x,t) + \big(\lambda + \mu_1(x)\big)P_{0,1}(x,t) = 0 \tag{6}$$

$$\frac{\partial}{\partial x}P_{n,2}(x,t) + \frac{\partial}{\partial t}P_{n,2}(x,t) + \big(\lambda + \mu_2(x)\big)P_{n,2}(x,t) = \lambda \sum_{k=1}^{n} \pi_k P_{n-k,2}(x,t) \tag{7}$$

$$\frac{\partial}{\partial x}P_{0,2}(x,t) + \frac{\partial}{\partial t}P_{0,2}(x,t) + \big(\lambda + \mu_2(x)\big)P_{0,2}(x,t) = 0 \tag{8}$$

$$\frac{\partial}{\partial x}V_n(x,t) + \frac{\partial}{\partial t}V_n(x,t) + \big(\lambda + \alpha(x)\big)V_n(x,t) = \lambda \sum_{k=1}^{n} \pi_k V_{n-k}(x,t) \tag{9}$$

$$\frac{\partial}{\partial x}V_0(x,t) + \frac{\partial}{\partial t}V_0(x,t) + \big(\lambda + \alpha(x)\big)V_0(x,t) = 0 \tag{10}$$

$$\frac{d}{dt}Q(t) + \lambda Q(t) = \int_0^\infty V_0(x,t)\alpha(x)dx \tag{11}$$

Equations (5) – (11) are to be solved subject to the following boundary conditions:

$$P_{n,1}(0,t) = \int_0^\infty V_{n+M}(t)\alpha(x)dx \tag{12}$$

$$P_{0,1}(0,t) = \sum_{b=1}^{M} \int_0^\infty V_b(x,t)\alpha(x)dx + \lambda Q(t) \tag{13}$$

$$P_{n,2}(0,t) = \int_0^\infty P_{n,1}(x,t)\mu_1(x)dx \tag{14}$$

$$V_n(0,t) = p\int_0^\infty P_{n-M,2}(x,t)\mu_2(x)dx + (1-p)\int_0^\infty P_{n,2}(x,t)\mu_2(x)dx \tag{15}$$

$$V_0(0,t) = (1-p)\int_0^\infty P_{0,2}(x,t)\mu_2(x)dx \tag{16}$$

We assume that initially the server is available but idle because of less than $M$ customers so that the initial conditions are

$$V_n(0) = 0;\ V_0(0) = 0;\ Q(0) = 1$$
$$P_{n,j}(0) = 0, \text{for } n = 0,1,2,\ldots \text{and } j = 1,2. \tag{17}$$

## V. PROBABILITY GENERATING FUNCTION OF THE QUEUE SIZE: THE TIME DEPENDENT SOLUTION

We define the following probability generating functions:

$$\left.\begin{aligned} P_j(x,z,t) &= \sum_{n=0}^{\infty} P_{n,j}(x,t)z^n,\ \ j = 1,2 \\ P_j(z,t) &= \sum_{n=0}^{\infty} P_{n,j}(t)z^n,\ j = 1,2 \\ V(z,t) &= \sum_{n=0}^{\infty} V_n(t)z^n \\ \pi(z) &= \sum_{n=1}^{\infty} \pi_n z^n \end{aligned}\right\} \tag{18}$$

Define the Laplace-Stieltjes Transform of a function $f(t)$ as follows:

$$\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt \tag{19}$$

Taking Laplace Transform of equations (5) – (11) and using (18), we get,

$$\frac{\partial}{\partial x}\bar{P}_{n,1}(x,s) + \big(s + \lambda + \mu_1(x)\big)\bar{P}_{n,1}(x,s) = \lambda \sum_{k=1}^{n} \pi_k \bar{P}_{n-k,1}(x,s) \tag{20}$$

$$\frac{\partial}{\partial x}\bar{P}_{0,1}(x,s) + \big(s + \lambda + \mu_1(x)\big)\bar{P}_{0,1}(x,s) = 0 \tag{21}$$

$$\frac{\partial}{\partial x}\bar{P}_{n,2}(x,s) + \big(s + \lambda + \mu_2(x)\big)\bar{P}_{n,2}(x,s) = \lambda \sum_{k=1}^{n} \pi_k \bar{P}_{n-k,2}(x,s) \tag{22}$$

$$\frac{\partial}{\partial x}\bar{P}_{0,2}(x,s) + \big(s + \lambda + \mu_2(x)\big)\bar{P}_{0,2}(x,s) = 0 \tag{23}$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s) + \big(s + \lambda + \alpha(x)\big)\bar{V}_n(x,s) = \lambda \sum_{k=1}^{n} \pi_k \bar{V}_{n-k}(x,s) \tag{24}$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s) + \big(s+\lambda+\alpha(x)\big)\bar{V}_0(x,s) = 0 \tag{25}$$
$$(s+\lambda)\bar{Q}(s) = 1 + \int_0^\infty \bar{V}_0(x,s)\alpha(x)dx \tag{26}$$
$$\bar{P}_{n,1}(0,s) = \int_0^\infty \bar{V}_{n+M}(x,s)\alpha(x)dx \tag{27}$$
$$\bar{P}_{0,1}(0,s) = \sum_{b=1}^{M}\int_0^\infty \bar{V}_b(x,s)\alpha(x)dx + \lambda\bar{Q}(s) \tag{28}$$
$$\bar{P}_{n,2}(0,s) = \int_0^\infty \bar{P}_{n,1}(x,s)\,\mu_1(x)dx \tag{29}$$
$$\bar{V}_n(0,s) = p\int_0^\infty \bar{P}_{n-M,2}(x,s)\mu_1(x)dx + (1-p)\int_0^\infty \bar{P}_{n,2}(x,s)\mu_2(x)dx \tag{30}$$
$$\bar{V}_n(0,s) = (1-p)\int_0^\infty \bar{P}_{0,2}(x,s)\mu_2(x)dx \tag{31}$$

Multiplying the equation (20) by $z^n$ and summing over $n$ from 1 to $\infty$, adding equation (21) and using the generating functions defined in (18), we obtain,

$$\frac{\partial}{\partial x}\bar{P}_1(x,z,s) + \{s+\lambda\big(1-\pi(z)\big)+\mu_1(x)\}\bar{P}_1(x,z,s) = 0 \tag{32}$$

Performing similar operations on equations (12) – (25), we obtain,

$$\frac{\partial}{\partial x}\bar{P}_2(x,z,s) + \{s+\lambda\big(1-\pi(z)\big)+\mu_2(x)\}\bar{P}_2(x,z,s) = 0 \tag{33}$$
$$\frac{\partial}{\partial x}\bar{V}(x,z,s) + \{s+\lambda\big(1-\pi(z)\big)+\alpha(x)\}\bar{V}(x,z,s) = 0 \tag{34}$$

Multiplying the equation (27) by $z^{n+M}$ and summing over $n$ from 1 to $\infty$ and adding, multiplying the equation (28) by $z^M$, and using the generating functions defined in (18) and using (26), we obtain,

$$\bar{P}_1(0,z,s) = z^{-M}\int_0^\infty \bar{V}(x,z,s)\alpha(x)dx + \sum_{b=1}^{M-1}(1-z^{-M+b})\int_0^\infty \bar{V}_b(x,s)\alpha(x)dx + [\lambda-(s+\lambda)z^{-M}]\bar{Q}(s) + z^{-M} \tag{35}$$

Multiplying the equation (29) by $z^n$ and summing over $n$ from 0 to $\infty$ and using the generating functions defined in (14), we obtain,

$$\bar{P}_2(0,z,s) = \int_0^\infty \bar{P}_1(x,z,s)\mu_1(x)dx \tag{36}$$

Multiplying the equation (30) by $z^{n-M}$ and summing over $n$ from 1 to $\infty$ and adding, multiplying the equation (31) by $z^{-M}$, and using the generating functions defined in (18), we obtain,

$$\bar{V}(0,z,s) = \{(1-p)+pz^M\}\int_0^\infty \bar{P}_2(x,z,s)\mu_2(x)dx \tag{37}$$

We now integrate equations (32) - (34) between the limits 0 and $x$ and obtain,

$$\bar{P}_1(x,z,s) = \bar{P}_1(0,z,s)e^{-Rx-\int_0^x \mu_1(x)dx} \tag{38}$$
$$\bar{P}_2(x,z,s) = \bar{P}_2(0,z,s)e^{-Rx-\int_0^x \mu_2(x)dx} \tag{39}$$
$$\bar{V}(x,z,s) = \bar{V}(0,z,s)e^{-Rx-\int_0^x \alpha(x)dx} \tag{40}$$

Where $R = s+\lambda\big(1-\pi(z)\big)$

Integrating equations (38) - (40) by parts, with respect to $x$, we get,

$$\bar{P}_1(z,s) = \bar{P}_1(0,z,s)\left[\frac{1-\bar{G}_1(R)}{R}\right] \tag{41}$$
$$\bar{P}_2(z,s) = \bar{P}_2(0,z,s)\left[\frac{1-\bar{G}_2(R)}{R}\right] \tag{42}$$
$$\bar{V}(z,s) = \bar{V}(0,z,s)\left[\frac{1-\bar{V}(R)}{R}\right] \tag{43}$$

Where $\bar{G}_j(R) = \int_0^\infty e^{-R\,x}dG_j(x)$, is the Laplace Transform of $j^{th}$ type of service, $j = 1,2$.
$\bar{V}(R) = \int_0^\infty e^{-R\,x}dV(x)$ is the Laplace Transform of vacation time.

Multiplying the equations (38), (39) and (40) by $\mu_1(x)$, $\mu_2(x)$and $\alpha(x)$ integrating by parts, with respect to $x$, we get,

$$\int_0^\infty \bar{P}_1(x,z,s)\mu_1(x)dx = \bar{P}_1(0,z,s)\bar{G}_1(R) \tag{44}$$
$$\int_0^\infty \bar{P}_2(x,z,s)\mu_2(x)dx = \bar{P}_2(0,z,s)\bar{G}_2(R) \tag{45}$$
$$\int_0^\infty \bar{V}(x,z,s)\alpha(x)dx = \bar{V}(0,z,s)\bar{V}(R) \tag{46}$$

Substituting (44) in (36) we get,

$$\bar{P}_2(0,z,s) = \bar{P}_1(0,z,s)\bar{G}_1(R) \tag{47}$$

Substituting (46) in (35) and using (37) & (47), we get,

$$\bar{P}_1(0,z,s) = \frac{\sum_{b=1}^{M-1}(z^M-z^b)\int_0^\infty \bar{V}_b(x,s)\alpha(x)dx + \big(\lambda z^M-(s+\lambda)\big)\bar{Q}(s)+1}{z^M-[(1-p)+pz^M]\bar{G}_1(R)\bar{G}_2(R)\bar{V}(R)} \tag{48}$$

 

Substituting (48) in (47) we get,

$$\bar{P}_2(0,z,s) = \left[\frac{\sum_{b=1}^{M-1}(z^M - z^b)\int_0^\infty \bar{V}_b(x,s)\alpha(x)dx + \left(\lambda z^M - (s+\lambda)\right)\bar{Q}(s)+1}{z^M - [(1-p)+pz^M]\bar{G}_1(R)\bar{G}_2(R)\bar{V}(R)}\right]\bar{G}_1(R) \tag{49}$$

Substituting (45) in (37) and using (49), we get,

$$\bar{V}(0,z,s) = \{(1-p)+pz^M\}\left[\frac{\sum_{b=1}^{M-1}(z^M - z^b)\int_0^\infty \bar{V}_b(x,s)\alpha(x)dx + \left(\lambda z^M - (s+\lambda)\right)\bar{Q}(s)+1}{z^M - [(1-p)+pz^M]\bar{G}_1(R)\bar{G}_2(R)\bar{V}(R)}\right]\bar{G}_1(R)\bar{G}_2(R) \tag{50}$$

Substituting (48), (49) and (50) in (41), (42) and (43) respectively, we get,

$$\bar{P}_1(z,s) = \left[\frac{\sum_{b=1}^{M-1}(z^M - z^b)\int_0^\infty \bar{V}_b(x,s)\alpha(x)dx + \left(\lambda z^M - (s+\lambda)\right)\bar{Q}(s)+1}{z^M - [(1-p)+pz^M]\bar{G}_1(R)\bar{G}_2(R)\bar{V}(R)}\right]\left[\frac{1-\bar{G}_1(R)}{R}\right] \tag{51}$$

$$\bar{P}_2(z,s) = \left[\frac{\sum_{b=1}^{M-1}(z^M - z^b)\int_0^\infty \bar{V}_b(x,s)\alpha(x)dx + \left(\lambda z^M - (s+\lambda)\right)\bar{Q}(s)+1}{z^M - [(1-p)+pz^M]\bar{G}_1(R)\bar{G}_2(R)\bar{V}(R)}\right]\left[\frac{1-\bar{G}_2(R)}{R}\right]\bar{G}_1(R) \tag{52}$$

$$\bar{V}(z,s) = \{(1-p)+pz^M\}\left[\frac{\sum_{b=1}^{M-1}(z^M - z^b)\int_0^\infty \bar{V}_b(x,s)\alpha(x)dx + \left(\lambda z^M - (s+\lambda)\right)\bar{Q}(s)+1}{z^M - [(1-p)+pz^M]\bar{G}_1(R)\bar{G}_2(R)\bar{V}(R)}\right]\left[\frac{1-\bar{V}(R)}{R}\right]\bar{G}_1(R)\bar{G}_2(R) \tag{53}$$

We note that there are $M$ unknowns, $\bar{Q}(s)$and $\bar{V}_b(x,s), b = 1,2,\ldots,M-1$ appearing in equations (51), (52) and (53).

Now (51), (52) and (53) gives the probability generating function of the service system with $M$ unknowns. By Rouche's theorem of complex variables, it can be proved that $z^M - [(1-p)+pz^M]\bar{G}_1(R)\bar{G}_2(R)\bar{V}(R)$ has $M$ zeroes inside the contour $|z| = 1$. Since $\bar{P}_1(z,s), \bar{P}_2(z,s)$and$\bar{V}(z,s)$ are analytic inside the unit circle $|z| = 1$, the numerator in the right hand side of equations(51), (52) and (53) must vanish at these points, which gives rise to a set of $M$ linear equations which are sufficient to determine $M$unknowns.

## VI. THE STEADY STATE RESULTS

To define the steady state probabilities and corresponding generating functions, we drop the argument $t$, and for that matter the argument $s$ wherever it appears in the time-dependent analysis up to this point. Then the corresponding steady state results can be obtained by using the well-known Tauberian Property

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) \tag{54}$$

If the limit on the right exists.

Now (51), (52) and (53) we have,

$$P_1(z) = \frac{\{U^* + \lambda(z^M - 1)Q\}\left\{\frac{1-\bar{G}_1(f_1(z))}{f_1(z)}\right\}}{z^M - [(1-p)+pz^M]\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\bar{V}(f_1(z))} \tag{55}$$

$$P_2(z) = \frac{\{U^* + \lambda(z^M - 1)Q\}\bar{G}_1(f_1(z))\left\{\frac{1-\bar{G}_2(f_1(z))}{f_1(z)}\right\}}{z^M - [(1-p)+pz^M]\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\bar{V}(f_1(z))} \tag{56}$$

$$V(z) = \frac{\{(1-p)+pz^M\}\{U^* + \lambda(z^M - 1)Q\}\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\left\{\frac{1-\bar{V}(f_1(z))}{f_1(z)}\right\}}{z^M - [(1-p)+pz^M]\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\bar{V}(f_1(z))} \tag{57}$$

The $M$ unknowns, $Q$ and $\int_0^\infty V_b(x)\alpha(x)dx$ , $b = 1,2,\ldots,M-1$can be determined as before.
Where $f_1(z) = \lambda(1-\pi(z))$ and $U^* = \sum_{b=1}^{M-1}(z^M - z^b)\int_0^\infty V_b(x)\alpha(x)dx$

Let $A_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system.

$$\text{i.e.}, A_q(z) = P_1(z) + P_2(z) + V(z) \tag{58}$$

Then adding equations (55), (56) and (57), we obtain,

$$A_q(z) = \frac{\{U^* + \lambda(z^M - 1)Q\}\left[\left\{\frac{1-\bar{G}_1(f_1(z))}{f_1(z)}\right\} + \bar{G}_1(f_1(z))\left\{\frac{1-\bar{G}_2(f_1(z))}{f_1(z)}\right\} + \{(1-p)+pz^M\}\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\left\{\frac{1-\bar{V}(f_1(z))}{f_1(z)}\right\}\right]}{z^M - [(1-p)+pz^M]\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\bar{V}(f_1(z))} \tag{59}$$

In order to find $Q$, we use the normalization condition

$A_q(1) + Q = 1$

$$\text{i.e.}, P_1(1) + P_2(1) + V(1) + Q = 1 \tag{60}$$

Note that for $z = 1$, $A_q(1) = P_1(1) + P_2(1) + V(1)$ is indeterminate of $\frac{0}{0}$ form.

 

Therefore, we apply L'Hôpital's Rule on (59), we get,

$$A_q(1) = \frac{E^*[U+\lambda MQ]}{M(1-p)-\lambda E(I)E^*} \tag{61}$$

We used $\bar{G}_j(0) = 1,\ j = 1,2,\ \bar{V}(0) = 1,\ \pi'(1) = E(I)$, where $I$ denotes the number of customers in an arriving batch and therefore, $E(I)$ is the mean of the batch size of the arriving customers. Similarly $E(S_1)$, $E(S_2)$ and $E(V)$ are the mean service times of type 1, type 2 services and vacation time, respectively.
Where $E^* = E(S_1) + E(S_2) + E(V)$ and $U = \sum_{b=1}^{M-1}(M-b)\int_0^\infty V_b(x)\alpha(x)dx$

Therefore, adding $Q$ to equation (61) and equating to 1 and simplifying we get,

$$Q = 1 - \frac{E^*(M\lambda+U)}{M(1-p)+\lambda[M-E(I)]E^*} \tag{62}$$

Equation (62) gives the probability that the server is idle.

From equation (62) the utilization factor, $\rho$ of the system is given by

$$\rho = \frac{E^*(M\lambda+U)}{M(1-p)+\lambda[M-E(I)]E^*} \tag{63}$$

Where $\rho < 1$ is the stability condition under which the steady state exists. Substituting for $Q$ from (62) into (59), we have completely and explicitly determined $A_q(z)$, the probability generating function of the queue size.

## VII. THE AVERAGE QUEUE SIZE AND THE SYSTEM SIZE

Let $L_q$ denote the mean number of customers in the queue under the steady state.

$$\text{Then } L_q = \frac{d}{dz}A_q(z)\Big|_{z=1} \tag{64}$$

Since the formula gives $\frac{0}{0}$ form, then we write $A_q(z)$ given in (59) as

$$A_q(z) = \frac{N(z)}{D(z)}$$

Where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of (59) respectively.

Then using L'Hôpital's Rule twice we obtain,

$$L_q = \lim_{z\to 1}\frac{D'(z)N''(z)-N'(z)D''(z)}{2\left(D'(z)\right)^2} \tag{65}$$

Where primes and double primes in (65) denote first and second derivatives at $z = 1$, respectively. Carrying out the derivatives at $z = 1$ we have,

$$N'(1) = E^*[U + \lambda MQ] \tag{66}$$

$$D'(1) = M(1-p) - \lambda E(I)E^* \tag{67}$$

$$N''(1) = E^*[U^{**} + \lambda M(M-1)Q] + (U + \lambda MQ)[\lambda E(I)E^{**} + 2\lambda E(I)E_1^* + 2E(V)Mp] \tag{68}$$

$$D''(1) = M(M-1) - \left[\lambda^2\left(E(I)\right)^2\{E^{**} + 2E_1^*\} + \lambda E\left(I(I-1)\right)E^* + pM\{2\lambda E(I)E^* + M - 1\}\right] \tag{69}$$

Therefore, the mean number of customers in the queue is

$$L_q = \frac{[M(1-p)-\lambda E(I)E^*]\left\{\begin{matrix} E^*[U^{**}+\lambda M(M-1)Q] \\ +(U+\lambda MQ)\begin{bmatrix}\lambda E(I)E^{**}+2\lambda E(I)E_1^* \\ +2E(V)Mp\end{bmatrix}\end{matrix}\right\} - [E^*\{U+\lambda MQ\}]\left\{M(M-1)-\begin{bmatrix}\lambda^2\left(E(I)\right)^2\{E^{**}+2E_1^*\}+\lambda E\left(I(I-1)\right)E^* \\ +pM\{2\lambda E(I)E^*+M-1\}\end{bmatrix}\right\}}{2[M(1-p)-\lambda E(I)E^*]^2} \tag{70}$$

Where $U^{**} = \sum_{b=1}^{M-1}\left(M(M-1) - b(b-1)\right)\int_0^\infty V_b(x)\alpha(x)dx$, $E^{**} = E(S_1^2) + E(S_2^2) + E(V^2)$ and
$E_1^* = E(S_1)E(S_2) + E(S_1)E(V) + E(S_2)E(V)$
Where $E\left(I(I-1)\right)$ is the second factorial moment of the batch size of the arriving customers. Similarly, $E(S_1^2), E(S_2^2)$ are the second moments of the service times of type 1, type 2 services, respectively. $E(V^2)$ is the second moment of the vacation time and $Q$ has been obtained in (62).

Further, the average number of customers in the system can be found as $L_s = L_q + \rho$ by using Little's formula.

## VIII. NUMERICAL RESULTS

By executing the numerical results, following arbitrary values are chosen:
$\lambda = 12,\ E(S_1) = 0.2,\ E(S_2) = 0.18,\ E(V) = 0.25, E(I) = 0.15,\ E\left(I(I-1)\right) = 0.01$ but the values are varied as below: $p = 0.1, 0.3, 0.4$ and $M = 10, 8, 4$. All the values are chosen in such a way that the steady state condition is satisfied.

**Table-1:** Computed values of the various characteristics of the system

| $M$ | $p$ | $1-p$ | $\rho$ | $Q$ | $L_q$ |
|---|---|---|---|---|---|
| 10 | 0.1 | 0.9 | 0.9266 | 0.0734 | 0.4777 |
| | 0.3 | 0.7 | 0.9494 | 0.0506 | 1.5156 |
| | 0.4 | 0.6 | 0.9612 | 0.0388 | 2.1389 |
| 8 | 0.1 | 0.9 | 0.9289 | 0.0711 | 0.4177 |
| | 0.3 | 0.7 | 0.9517 | 0.0483 | 1.3244 |
| | 0.4 | 0.6 | 0.9636 | 0.0364 | 1.8915 |
| 4 | 0.1 | 0.9 | 0.9404 | 0.0596 | 0.3592 |
| | 0.3 | 0.7 | 0.9640 | 0.0360 | 1.1149 |
| | 0.4 | 0.6 | 0.9763 | 0.0237 | 1.7188 |

The computed values of the various states of the server are shown in table 1; the utilization factor, the proportion of the idle time, average number of customers in the queue. It is shown that, for the different values of $p$ (probability of the feedback customer is increases), the server idle time is decreased, the utilization factor and the average number of customers in the queue of our model are increased.

## IX. CONCLUSION

The single server bulk queue with feedback, two choices of service and compulsory vacation is discussed. The transient solution, steady state results, the average number of customers in the queue and numerical results of the system are briefed in this paper.

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