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A NOTE ON COLOR ENERGY AND COLOR LAPLACIAN ENERGY OF GRAPHS

SHIGEHALLI V. S.
P. G. Department of Mathematics and Computing Sciences,
Rani Channamma University Belagavi, Karnataka, India.

KENCHAPPA S. BETAGERI*
Research Scholar, P.G. Studies in Dept. of Mathematics
Karnatak University Dharwad Karnataka, India.

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ABSTRACT

We have defined color Laplacian energy of graphs and proved many results about color Laplacian energy and established relationships between color eigenvalues, color Laplacian eigenvalues of a graphs in [5]. In this paper we obtained new lower bonds for color energy and color Laplacian energy of graphs and obtained various bounds for color eigenvalues and color Laplacian eigenvalues of G. We also obtained a bound for i.e., $\left|LE_\chi(G-e)-LE_\chi(G)\right|<4$ and 4 is the best possible bound.

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1.0 INTRODUCTION

Recently C. Adiga and et.al., have introduced and investigated many properties and also found many results on color energy and color eigenvalues of a graph in [1]. A coloring of graph G is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph G is called chromatic number and is denoted by $\chi(G)$. They noticed that, in general optimal coloring with $\chi(G)$ colors is not unique so the color energy $E_\chi(G)$ may be different for different optimal colorings. On the other hand, there do exist some uniquely colorable graphs, for example, the unitary Cayley graph X_n has a unique optimal coloring, thus its color energy with respect to minimum color is unique. They also established explicit formulae for color energy of unitary Cayley graph X_n and its complements.

Motivated by [1], we have defined color Laplacian energy in [5] and studied some results and bounds for the color Laplacian eigenvalues and color Laplacian energy of graphs. P.G Bhat and Sabitha D'Souza are also defined the same concept of color Laplacian energy of graphs independently see [4]. In this paper we will find further bounds for color energy and color Laplacian energy of graphs, in terms of color eigenvalues and color Laplacian eigenvalues. We used Polya-Szego Inequality and Ozekis Inequality to obtain some new lower bonds for color energy and color Laplacian energy of graphs and obtained various bounds for color eigenvalues and color Laplacian eigenvalues. We also obtained a bound for i.e., $\left|LE_\chi(G-e)-LE_\chi(G)\right|<4$ and 4 is the best possible bound.

This paper is organized as follows, in 1.0 we present short introduction of necessary definitions and basic results and in 1.1 we prove some theorems regarding color Laplacian eigenvalues. In 1.2, we present new bounds for color energy and color Laplacian energy in terms of color eigenvalues and color Laplacian eigenvalues of graphs.

Definition 1 [1]: (Color Matrix): The color matrix $A_c(G)=[a_{ij}]$ of G is a square matrix of order whose (i,j)-entries are as follows

$$[a_{ij}]=\begin{cases}1, & if\ v_i\ and\ v_j\ are\ adjacent\ with\ c(v_i)\neq c(v_j)\\ -1, & if\ v_i\ and\ v_j\ are\ not\ adjacent\ with\ c(v_i)=c(v_j)\\ 0, & otherwise\end{cases}$$

*Corresponding Author: Kenchappa S Betageri**

In the case that, G is simple graph of order n and m, the color matrix $A_c(G)$ will be $(-1, 0, 1)$ matrix with respect to a given coloring. Suppose that

$$P_c(G, \lambda) = \det\big(\lambda I - A_c(G)\big) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \cdots \ldots . + a_0$$

is the characteristic polynomial of $A_c(G)$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the color eigenvalues of G, i.e., the roots of $P_c(G, \lambda)$. The color spectrum of G is defined as the set of color eigenvalues of $A_c(G)$ together with their multiplicities. If the graph G is colored with $\chi(G)$ colors, then the color energy of a graph with respect $\chi(G)$ is denoted by $E_\chi(G)$. This is called chromatic energy of G. The color energy of G is the sum of the absolute values of its color eigenvalues.

Definition 2 [5]: (Color Laplacian Matrix)**:** Let G be a simple colored graph. We denote the diagonal matrix with the degrees as diagonal elements by$D(G)$, $A_c(G)$ is color matrix of graph G. The color Laplacian matrix is $L_c(G) = D(G) - A_c(G)$, we can also write Color Laplacian Matrix as

$$L_c(G) = [l_{ij}] = \begin{cases} d(v_i), & if\ v_i = v_j \\ -1, & if\ v_i\ and\ v_j\ are\ adjacent\ with\ c(v_i) \neq c(v_j) \\ 1\ , & if\ v_i\ and\ v_j\ are\ not\ adjacent\ with\ c(v_i) = c(v_j) \\ 0, & otherwise \end{cases}$$

Suppose that

$$P_c(G, \mu\ ) = \det\big(\mu I - L_c(G)\big) = a_0\mu^n + a_1\mu^{n-1} + a_2\mu^{n-2} + a_3\mu^{n-3} + \cdots \ldots . + a_0$$

is the characteristic polynomial of $L_c(G)$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the color eigenvalues of G, i.e., the roots of $P_c(G, \mu)$. The color Laplacian spectrum of G is defined as the set of color Laplacian eigenvalues of $L_c(G)$ together with their multiplicities. If the graph G is colored with $\chi(G)$ colors, then the color Laplacian matrix of a graph with respect $\chi(G)$ is denoted by $L_\chi(G)$. This is called chromatic energy of G. We noticed an important property of color Laplacian matrix $L_\chi(G)$ is not positive semi-definite matrix in general, but some special class of graphs whose color Laplacian matrix $L_\chi(G)$ is positive semi-definite matrix, viz., $K_n, K_{n,m}, S_n, W_n\ and\ C_n$.

Now we define color Laplacian energy of G.

Definition 3 [5]: (Color Laplacian Energy**):** If G is an (n, m)– graph and its color Laplacian eigenvalues are $\mu_{1,}\, \mu_2, \ldots\ \ \ldots, \mu_n$, then the color Laplacian energy of G denoted by $LE_c(G) = \sum_{i=1}^n |\gamma_i|$, where $\gamma_i = \mu_i - \frac{2m}{n}$.

i.e., $LE_c(G) = \sum_{i=1}^n \left|\mu_i - \frac{2m}{n}\right|$

If G colored with $\chi(G)$ colors, then chromatic Laplacian matrix of G, is denoted by $L_\chi(G)$ and the energy of graph with respect $\chi(G)$ is called chromatic Laplacian energy and is denoted by $LE_\chi(G)$.

We have established some properties of color Laplacian eigenvalues and color Laplacian energy in [5]. The color eigenvalues and color Laplacian eigenvalues obey the following well-known relations respectively.

$$\sum_{i=1}^n \lambda_i = 0; \quad \sum_{i=1}^n \lambda_i^2 = 2(m + m_c'), \quad \sum_{i=1}^n \mu_i = 2m;$$
$$\sum_{i=1}^n \mu_i^2 = 2M_1 \quad \sum_{i=1}^n \gamma_i = 0; \quad \sum_{i=1}^n \gamma_i^2 = 2M_2$$

Where $M_1 = [(m + m_c') + \frac{1}{2}\sum_{i=1}^n d_i^2]$, $M_2 = [(m + m_c') + \frac{1}{2}\left(\sum_{i=1}^n d_i^2 - \frac{4m^2}{n}\right)]$ and m_c' is the number of pairs of non adjacent vertices receiving the same color in G.

Further if G is regular of degree k then we have proved the theorem in [5].

Theorem 4 [1]: If the graph is $k -$ color regular graph, then $LE_\chi(G) = E_\chi(G)$.

1.1 MAIN RESULTS

In this section we prove some new theorems, before proving Theorem 5 we note that the interlacing theorem for the color Laplacian matrix cannot be true. This fact is similar to the case of Laplacian eigenvalues and normalized Laplacian eigenvalues of graphs, i.e., when we delete a colored vertex in G, it affects its neighbor colored vertices in G. Nevertheless, we can state another interlacing theorem for the color Laplacian eigenvalues.

Theorem 5: Let G be a graph and $e \in E(G)$. Let $(G - e)$ be the graph obtained by deleting e with $\chi(G) = \chi(G - e)$. If $\mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots\ \leq \mu_n$ be the color Laplacian eigenvalues G. Then

$$\mu_1(G) \leq \mu_1(G - e) \leq \mu_2(G) \leq \mu_2(G - e) \leq \cdots\ \mu_{n-1}(G - e) \leq \mu_n(G)$$

Theorem 6: Let G be a colored graph of order n and $(G - e)$ is graph obtained by deleting an edge. Then $\mu_i - \mu_i{}' \geq 0$ and $\sum_{i=1}^n [\mu_i - \mu_i{}'] = 2$. Where μ_i are the color Laplacian eigenvalues of G and $\mu_i{}'$ are the color Laplacian eigenvalues of $(G - e)$.

Proof: First part of the proof is straightforward by interlacing property of color Laplacian eigenvalues or by Theorem 5. Second part of the proof is follows from the fact that, $\sum_{i=1}^{n} \mu_i = 2\,m$ and $\sum_{i=1}^{n} \mu_i{}' = 2(m-1)$. Therefore, $\sum_{i=1}^{n}[\mu_i - \mu_i{}'] = 2$.

Now we have an immediate theorem, which gives the best bound for the difference of color Laplacian energy of G and color Laplacian energy of G $(G-e)$ follows from Theorem 6 above.

Theorem 7: Suppose G is a colored graph with $\chi(G)$. Then $\left|LE_\chi(G-e) - LE_\chi(G)\right| < 4$ and 4 is the best possible bound.

Proof: Define $\mu_i{}' = \mu_i(G-e)$ with $\chi(G-e) = \chi(G)$. We noticed that $\mu_i - \mu_i{}' \geq 0$ and $\sum_{i=1}^{n}[\mu_i - \mu_i{}'] = 2$. So, there exist $i, 1 \leq i \leq n$, such that $\mu_i > \mu_i{}'$. This implies that

$$\sum_{i=1}^{n}\left|\mu_i - \mu_i{}' - \frac{2}{n}\right| < \sum_{i=1}^{n}\left|\mu_i - \mu_i{}'\right| + \frac{2}{n}$$

and we have

$$\begin{aligned}
\left|LE_\chi(G-e) - LE_\chi(G)\right| &= \left|\sum_{i=1}^{n}\left|\mu_i - 2\frac{m}{n}\right| - \left|\mu_i{}' - 2\frac{m-1}{n}\right|\right| \\
&= \left|\sum_{i=1}^{n}\left[\left|\mu_i - 2\frac{m}{n}\right| - \left|\mu_i{}' - 2\frac{m-1}{n}\right|\right]\right| \\
&\leq \sum_{i=1}^{n}\left|\left|\mu_i - 2\frac{m}{n}\right| - \left|\mu_i{}' - 2\frac{m-1}{n}\right|\right| \\
&\leq \sum_{i=1}^{n}\left|\mu_i - \mu_i{}' - \frac{2}{n}\right| \\
&< \sum_{i=1}^{n}\left|\mu_i - \mu_i{}'\right| + \frac{2}{n} \\
&= \sum_{i=1}^{n}\left[\mu_i - \mu_i{}' + \frac{2}{n}\right] = 4.
\end{aligned}$$

To complete the argument we construct a sequence of $\{G_n\}_{n\geq 2}$ of colored graphs such that $\left|LE_\chi(G-e) - LE_\chi(G)\right| \to 4$. Define $G_n = \overline{K_n} + e$ with $\chi(G) = n$. Then $LE_\chi(G_n) = 4 - \frac{4}{n}$ and $LE_\chi(G_n - e) = 0$ and so $\left|LE_\chi(G-e) - LE_\chi(G)\right| = 4 - \frac{4}{n} \to 4$. This completes the argument.

1.1 BONDS FOR COLOR ENERGY AND COLOR LAPLACIAN ENERGY OF GRAPHS

In this section we obtain some bounds for the color energy and color Laplacian energy in terms of color eigenvalues and color Laplacian eigenvalues of graphs. In order to obtain these bounds we require the smallest color eigenvalue and smallest color Laplacian eigenvalue are both must be non zeros. So we assume that $|\lambda_n|$ and $|\mu_n|$ are smallest color eigenvalue and smallest color Laplacian eigenvalue of G.

For the sake of completeness we mention below two preparatory results which are help us to prove following bounds.

Theorem 8 [2]: (Polya-Szego Inequality): Suppose a_i and b_i, $1 \leq i \leq n$ are positive real numbers, then

$$\sum_{i=1}^{n} \mathrm{a}_\mathrm{i}^2 \sum_{i=1}^{n} \mathrm{b}_\mathrm{i}^2 \leq \frac{1}{4}\left(\sqrt{\frac{P_1P_2}{p_1p_2}} + \sqrt{\frac{p_1p_2}{P_1P_2}}\right)^2 \left(\sum_{i=1}^{n} \mathrm{a}_\mathrm{i}\ \mathrm{b}_\mathrm{i}\right)^2$$

Where $P_1 = \max_{1\leq i\leq n} a_i$; $P_2 = \max_{1\leq i\leq n} b_i$ and $p_1 = \min_{1\leq i\leq n} a_i$; $p_2 = \min_{1\leq i\leq n} b_i$

Theorem 9 [3]: (Ozekis Inequality): If a_i and b_i, $1 \leq i \leq n$ are non negative real numbers, then

$$\sum_{i=1}^{n} \mathrm{a}_\mathrm{i}^2 \sum_{i=1}^{n} \mathrm{b}_\mathrm{i}^2 - \left(\sum_{i=1}^{n} \mathrm{a}_\mathrm{i}\ \mathrm{b}_\mathrm{i}\right)^2 \leq \frac{n^2}{4}(P_1P_2 - p_1p_2)^2$$

Where P_i and p_i are defined similar to theorem (8).

At first we apply Polya-Szego Inequality to obtain a simple inequality on the color eigenvalues and color energy of graphs. By theorem (8) we have the following theorem.

Theorem 10: Let G be a colored graph with λ_n and λ_1 are the smallest and largest color eigenvalues of G respectively. Then

$$LE(G) = E_\chi(G) \geq \frac{2}{|\lambda_1| + |\lambda_n|}\sqrt{2n(m+m_c')}\sqrt{(|\lambda_1||\lambda_n|)}$$

Proof: Suppose $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the color eigenvalues of $A_\chi(G)$. We also assume that $a_i = |\lambda_i|$ and $b_i = 1$ for $i = 1,2,\ldots n$. Then by applying theorem (8), we have

$$\sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} 1^2 \leq \frac{1}{4}\left(\sqrt{\frac{|\lambda_1|}{|\lambda_n|}} + \sqrt{\frac{|\lambda_n|}{|\lambda_1|}}\right)^2 \left(\sum_{i=1}^{n}|\lambda_i|\right)^2$$

$$n\sum_{i=1}^{n} |\lambda_i|^2 \leq \frac{1}{4}\left(\sqrt{\frac{|\lambda_1|}{|\lambda_n|}} + \sqrt{\frac{|\lambda_n|}{|\lambda_1|}}\right)^2 \left(\sum_{i=1}^{n}|\lambda_i|\right)^2$$

Since $\sum_{i=1}^{n} |\lambda_i|^2 = \sum_{i=1}^{n} {\lambda_i}^2 = 2(m+m_c')$ and $\sum_{i=1}^{n}|\lambda_i| = E_\chi(G)$ therefore we have,

$$2n(m+m_c') \leq \frac{1}{4}\left(\frac{|\lambda_1|}{|\lambda_n|} + \frac{|\lambda_n|}{|\lambda_1|} + 2\sqrt{\frac{|\lambda_1|}{|\lambda_n|}} \times \sqrt{\frac{|\lambda_n|}{|\lambda_1|}}\right)\left(E_\chi(G)\right)^2$$

$$8n(m+m_c') \leq \left(\frac{|\lambda_1|}{|\lambda_n|} + \frac{|\lambda_n|}{|\lambda_1|} + 2\right)\left(E_\chi(G)\right)^2$$

$$8n(m+m_c') \leq \left(\frac{|\lambda_1|^2 + |\lambda_n|^2 + 2|\lambda_1||\lambda_n|}{|\lambda_1||\lambda_n|}\right)\left(E_\chi(G)\right)^2$$

$$8n(m+m_c')(|\lambda_1||\lambda_n|) \leq (|\lambda_1| + |\lambda_n|)^2\left(E_\chi(G)\right)^2$$

$$\frac{8n(m+m_c')(|\lambda_1||\lambda_n|)}{(|\lambda_1| + |\lambda_n|)^2} \leq \left(E_\chi(G)\right)^2$$

$$\frac{2}{|\lambda_1|+|\lambda_n|}\sqrt{2n(m+m_c')}\sqrt{(|\lambda_1||\lambda_n|)} \leq E_\chi(G).$$

To obtain next inequality, we need to apply theorem (9) we get

Theorem 11: Let G be a colored graph with λ_n and λ_1 are the smallest and largest color eigenvalues of G respectively. Then

$$LE(G) = E_\chi(G) \geq \sqrt{2n(m+m_c') - \frac{n^2(|\lambda_1| - |\lambda_n|)^2}{4}}$$

Proof: Suppose $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the color eigenvalues of $A_\chi(G)$. We also assume that $a_i = |\lambda_i|$ and $b_i = 1$ for $i = 1,2,\ldots n$. Then by applying Theorem 9, we have

$$\sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} 1^2 - \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \leq \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2$$

$$n\sum_{i=1}^{n} |\lambda_i|^2 - \left(E_\chi(G)\right)^2 \leq \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2$$

Since $|\lambda_i|^2 = {\lambda_i}^2 = 2(m+m_c')$ and $\sum_{i=1}^{n}|\lambda_i| = E_\chi(G)$ therefore we have,

$$2n(m+m_c') - \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2 \leq \left(E_\chi(G)\right)^2$$

$$\sqrt{2n(m+m_c') - \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2} \leq E_\chi(G).$$

Further to prove second part in theorem 10 and theorem 11, by theorem 4, $E_\chi(G) = LE_\chi(G)$ for k −regular colored graphs.

Next we discuss same bounds for some non regular colored graphs using theorems (8) and (9).

Theorem 12: Let G be a colored graph with μ_1 and μ_n are the smallest and largest color Laplacian eigenvalues respectively. Then

$$\left(\sqrt{\frac{|\mu_n|}{|\mu_1|}}+\sqrt{\frac{|\mu_1|}{|\mu_n|}}\right)\geq 2\sqrt{\frac{nM_1}{M_1+(m+m_c^{'})}}$$

Proof: Suppose $\mu_1, \mu_2, \ldots \ldots, \mu_n$ are the color Laplacian eigenvalues of $L_\chi(G)$. We also assume that $a_i = 1$ and $b_i = |\mu_i|$ for $i = 1,2,\ldots,n$. Then by applying theorem (8), we have

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\mu_i|^2 \leq \frac{1}{4}\left(\sqrt{\frac{|\mu_n|}{|\mu_1|}}+\sqrt{\frac{|\mu_1|}{|\mu_n|}}\right)^2 \left(\sum_{i=1}^{n}|\mu_i|\right)^2$$

$$n\sum_{i=1}^{n} \mu_i^{\,2} \leq \frac{1}{4}\left(\sqrt{\frac{|\mu_n|}{|\mu_1|}}+\sqrt{\frac{|\mu_1|}{|\mu_n|}}\right)^2 \left(\sum_{i=1}^{n}|\mu_i|\right)^2$$

$$2n(m+m_c^{'})+\sum_{i=1}^{n} d_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{|\mu_n|}{|\mu_1|}}+\sqrt{\frac{|\mu_1|}{|\mu_n|}}\right)^2 \left(\sum_{i=1}^{n}|\mu_i|\right)^2 \ldots\ldots \quad (*)$$

From (*) we have

$$\begin{aligned}\left(\sum_{i=1}^{n}|\mu_i|\right)^2 &= \left(\sum_{i=1}^{n}|\mu_i|\right)\left(\sum_{j=1}^{n}|\mu_j|\right)\\ &= \sum_{i=1}^{n}|\mu_i|^2 + 2\sum_{i<j=1}^{n}|\mu_i||\mu_j|\\ &\geq \sum_{i=1}^{n}\mu_i^{\,2} + 2\left|\sum_{i\neq j=1}^{n}\mu_i\mu_j\right|\\ &\geq 2(m+m_c^{'})+\sum_{i=1}^{n} d_i^2 + 2(m+m_c^{'})\end{aligned}$$

Therefore (*) becomes

$$2n(m+m_c^{'})+\sum_{i=1}^{n} d_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{|\mu_n|}{|\mu_1|}}+\sqrt{\frac{|\mu_1|}{|\mu_n|}}\right)^2 2\Big(M_1+(m+m_c^{'})\Big)$$

$$\frac{2n(m+m_c^{'})+\sum_{i=1}^{n} d_i^2}{2\big(M_1+(m+m_c^{'})\big)} \leq \frac{1}{4}\left(\sqrt{\frac{|\mu_n|}{|\mu_1|}}+\sqrt{\frac{|\mu_1|}{|\mu_n|}}\right)^2$$

$$\frac{2nM_1}{2\big(M_1+(m+m_c^{'})\big)} \leq \frac{1}{4}\left(\sqrt{\frac{|\mu_n|}{|\mu_1|}}+\sqrt{\frac{|\mu_1|}{|\mu_n|}}\right)^2$$

$$4\left(\frac{nM_1}{\big(M_1+(m+m_c^{'})\big)}\right) \leq \left(\sqrt{\frac{|\mu_n|}{|\mu_1|}}+\sqrt{\frac{|\mu_1|}{|\mu_n|}}\right)^2$$

$$2\sqrt{\left(\frac{nM_1}{\big(M_1+(m+m_c^{'})\big)}\right)} \leq \left(\sqrt{\frac{|\mu_n|}{|\mu_1|}}+\sqrt{\frac{|\mu_1|}{|\mu_n|}}\right)$$

This completes the proof.

Theorem 13: Let G be a colored graph with μ_1 and μ_n are the smallest and largest color Laplacian eigenvalues respectively. Then

$$|\mu_n| - |\mu_{1,}| \geq \frac{2}{n}\sqrt{2[(n-1)\,M_1 - (m+m_c^{'})]}$$

Proof: Let $\mu_1, \mu_2, \ldots \quad \mu_n$ are the color Laplacian eigenvalues of G. Suppose $a_i = 1$ and $b_i = |\mu_i|$ for $i = 1,2,\ldots,n$. Then by applying theorem (9), we have

$$n\sum_{i=1}^{n} |\mu_i|^2 - \left(\sum_{i=1}^{n}|\mu_i|\right)^2 \leq \frac{n^2}{4}(|\mu_n| - |\mu_1|)^2$$

$$n\sum_{i=1}^{n}\mu_i{}^2-\sum_{i=1}^{n}\mu_i{}^2-2\sum_{i<j=1}^{n}|\mu_i||\mu_j| \leq \frac{n^2}{4}(|\mu_n|-|\mu_1|)^2$$

$$(n-1)\sum_{i=1}^{n}\mu_i{}^2-2\left|\sum_{i\neq j=1}^{n}\mu_i\mu_j\right| \leq \frac{n^2}{4}(|\mu_n|-|\mu_1|)^2$$

$$(n-1)[2(m+m_c^{'})+\sum_{i=1}^{n}d_i^2]-2(m+m_c^{'}) \leq \frac{n^2}{4}(|\mu_n|-|\mu_1|)^2$$

$$2(n-1)M_1-2(m+m_c^{'}) \leq \frac{n^2}{4}(|\mu_n|-|\mu_1|)^2$$

$$2[(n-1)M_1-(m+m_c^{'})] \leq \frac{n^2}{4}(|\mu_n|-|\mu_1|)^2$$

$$\frac{2}{n}\sqrt{2[(n-1)M_1-(m+m_c^{'})]} \leq |\mu_n|-|\mu_1|.$$

This completes the proof.

CONCLUSION

In this paper we obtained some new lower bonds for color energy and color Laplacian energy of graphs and obtained various bounds for color eigenvalues and color Laplacian eigenvalues. We also obtained a bound for i.e., $\left|LE_\chi(G-e)-LE_\chi(G)\right| < 4$ and 4 is the best possible bound.

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