International Journal of Mathematical Archive-7(10), 2016, 175-184
IJMA Available online through www.ijma.info ISSN 2229 – 5046

SOME COMMON FIXED POINT THEOREMS IN NON NEWTONIAN METRIC SPACE

ASHA RANI[1], KUMARI JYOTI[2]*

[1,2]Department of Mathematics, SRM University, Haryana, Sonepat- 131001, India.

(Received On: 08-09-16; Revised & Accepted On: 18-10-16)

ABSTRACT

In this paper, we have introduced the commutative, weakly commutative, compatible and weakly compatible maps in the setting of non Newtonian metric spaces. Also, some common fixed point theorems are proved for these mappings.

***KeyWords**: Non Newtonian calculus, commutative maps, weakly commutative maps, compatible maps, weakly compatible maps, common fixed point therorems.*

***Subject Classification**: 46S99, 54E40, 54H25.*

1. INTRODUCTION

The dawn of the fixed point theory starts when in 1912 Brouwer proved a fixed point result for continuous self maps on a closed ball. In 1922, Banach [2] gave a very useful result known as the Banach Contraction Principle. After which a lot of implications of Banach contraction came into existence ([1, 5, 6, 13, 14]).

A major shift in the arena of fixed point theory came in 1976 when Jungck [9], defined the concept of commutative maps and proved the common fixed point results for such maps. After which, Sessa[17] gave the concept of weakly compatible, and Jungck ([10,11])gave the concepts of compatibility and weak compatibility. Certain altercations of commutativity and compatibility can also be found in [7, 12, 15, 16, 18].

The study of non Newtonian calculi have been started in 1972 by Grossman and Katz [8]. These provide an alternative to the classical calculus and they include the geometric, anageometric and bigeometric calculi, etc. In 2002 Cakmac and Basar [4], have introduced the concept of non Newtonian metric space. Also they have given the triangle and Minkowski's inequalities in the sense of non-Newtonian calculus. Recently, Binbasioglu, *et al.* [3] discussed some topological properties of the non Newtonian metric space and also introduced the concept of fixed point theory for the non Newtonian Metric Space. The non-Newtonian calculi are alternatives to the classical calculus of Newton and Leibnitz. They provide a wide variety of mathematical tools for use in science, engineering and mathematics.

2. PRELIMINARIES

Now, we define the non-Newtonian real field and we give the relevant properties due to Cakmak and Basar [4].

A *generator* is defined as an injective function with domain $\mathbb{R}$ and the range of a generator is a subset of $\mathbb{R}$. Each generator generates one arithmetic if and only if each arithmetic is generated by one generator.

Let β be an exponential function defined as

$$\beta: \mathbb{R} \to \mathbb{R}^+, x \longmapsto \beta(x) = e^x = y,$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Suppose that this function β is a generator, that is, if $\beta = I, I(x) = x\ \forall\ x \in \mathbb{R}$, then β generates the classical arithmetic.

If β is an exponential function, then β generates geometrical arithmetic.

Define the set $\mathbb{R}(N)$ as

$$\mathbb{R}(N) \coloneqq \{\beta(x): x \in \mathbb{R}\},$$

Where $\mathbb{R}(N)$ is the set of non-Newtonian real numbers.

*Corresponding Author: Kumari Jyoti**

All concepts of β-arithmetic have similar properties in classical arithmetic. β-zero, β-one and all β-integers are formed as

$$\dots, \beta(-1), \beta(0), \beta(1), \dots.$$

Take any generator β with range A. Then define the operations β-addition, β-subtraction, β-multiplication, β-division and β-order in the following way for $x, y \in \mathbb{R}$, respectively:

β-addition $x \dot{+} y = \beta\{\beta^{-1}(x) + \beta^{-1}(y)\}$,
β –subtraction $x \dot{-} y = \beta\{\beta^{-1}(x) - \beta^{-1}(y)\}$,
β –multiplication $x \dot{\times} y = \beta\{\beta^{-1}(x) \times \beta^{-1}(y)\}$,
β –division $x \dot{/} y = \beta\{\beta^{-1}(x) \div \beta^{-1}(y)\}$,
β –order $x \dot{<} y \Leftrightarrow \beta(x) < \beta(y)$.

Proposition 2.1 [4]: $(\mathbb{R}(N), \dot{+}, \dot{\times})$ is a complete field.

For $x \in A \subset \mathbb{R}(N)$, a number β-square is described by $x \dot{\times} x$ and denoted by x^{2_N}. The symbol $\sqrt{x}^N$ denotes

$$t = \beta\left\{\sqrt{\beta^{-1}(x)}\right\}$$

which is the unique β nonnegative number whose β-square is equal to x and which means $t^{2_N} = x$, for each β nonnegative number t. Throughout this paper, x^{p_N} denotes the pth non-Newtonian exponent. Thus we have

$$x^{p_N} = x^{(p-1)_N} \dot{\times} x = \beta\{[\beta^{-1}(x)]^p\},$$

We denote by $|x|_N$ the β-absolute value of a number $x \in A \subset \mathbb{R}(N)$ defined as $\beta(|\beta^{-1}(x)|)$ and also

$$\sqrt{x^{2_N}}^N = |x|_N = \beta\{|\beta^{-1}(x)|\}$$

Thus,

$$|x|_N = \begin{cases} x, & x \dot{>} \beta(0), \\ \beta(0), & x = \beta(0), \\ \beta(0) \dot{-} x, & x \dot{<} \beta(0). \end{cases}$$

For $x_1, x_2 \in A \subseteq \mathbb{R}(N)$, the non-Newtonian distance $|\cdot|_N$ is defined as

$$|x_1 \dot{-} x_2|_N = \beta\{|\beta^{-1}(x_1) - \beta^{-1}(x_2)|\}.$$

This distance is commutative; i.e., $|x_1 \dot{-} x_2|_N = |x_2 \dot{-} x_1|_N$.

Take any $z \in \mathbb{R}(N)$, if $z \dot{>} \beta(0)$, then z is called a positive non-Newtonian real number; if $z \dot{<} \beta(0)$, then z is called a non-Newtonian negative real number and if $z = \beta(0)$, then z is called an unsigned non-Newtonian real number. Non-Newtonian positive real numbers are denoted by $\mathbb{R}^+(N)$ and non-Newtonian negative real numbers by $\mathbb{R}^-(N)$ [4].

The fundamental properties provided in the classical calculus are provided in non-Newtonian calculus, too.

Proposition 2.2 [4]: $|x \dot{\times} y|_N = |x|_N \dot{\times} |y|_N$ for any $x, y \in \mathbb{R}(N)$.

Proposition 2.3 [4]: The triangle inequality with respect to non-Newtonian distance $|\cdot|_N$, for any $x, y \in \mathbb{R}(N)$ is given by $|x \dot{+} y|_N \leq |x|_N \dot{+} |y|_N$.

The non-Newtonian metric spaces provide an alternative to the metric spaces introduced in [4].

Definition 2.4[4]: Let $X \neq \emptyset$ be a set. If a function $d_N: X \times X \to \mathbb{R}^+(N)$ satisfies the following axioms for all $x, y, z \in X$:
(NM1) $d_N(x, y) = \beta(0) = \dot{0}$ if and only if $x = y$,
(NM2) $d_N(x, y) = d_N(y, x)$,
(NM3) $d_N(x, y) \dot{\leq} d_N(x, z) \dot{+} d_N(z, y)$,
then it is called a non-Newtonian metric on X and the pair (X, d_N) is called a non-Newtonian metric space.

Proposition 2.5 [4]: Suppose that the non-Newtonian metric d_N on $\mathbb{R}(N)$ is such that $d_N(x, y) = |x \dot{-} y|_N$ for all $x, y \in \mathbb{R}(N)$, then $(\mathbb{R}(N), d_N)$ is a non-Newtonian metric space.

Definition 2.6 [4]: Let X be a vector space on $\mathbb{R}(N)$. If a function $\|\cdot\|_N : X \to \mathbb{R}^+(N)$ satisfies the following axioms for all $x, y \in X$ and $\lambda \in \mathbb{R}(N)$:
(NN1) $\|\cdot\|_N = \dot{0} \Leftrightarrow x = \dot{0}$,
(NN2) $\|\lambda \dot{\times} x\|_N = |\lambda|_N \dot{\times} \|x\|_N$,
(NN3) $\|x \dot{+} y\|_N \leq \|x\|_N \dot{+} \|y\|_N$,
then it is called a non-Newtonian norm on X and the pair $(X, \|\cdot\|_N)$ is called a non-Newtonian normed space.

Remark 2.7 [4]: Here it is easily seen that every non-Newtonian norm $\|\cdot\|_N$ on X produces a non-Newtonian metric d_N on X given by $d_N(x,y) = \|x \dot{-} y\|_N,\ for\ all\ x, y \in X$.

Now, we define some topological structures related to non-Newtonian metric spaces.

Proposition 2.8 [3]: Let (X, d_N) be a non-Newtonian metric space. Then we have the following inequality:
$|d_N(x,z) \dot{-} d_N(y,z)|_N \ \dot{\leq}\ d_N(x,y)$ for all $x, y, z \in X$.

Definition 2.9 [3]: Let (X, d_N) be a non-Newtonian metric space, $x \in X$ and $\varepsilon \dot{>} \dot{0}$, we now define a set $B_\varepsilon^N(x) = \{y \in X : d_N(x,y) \dot{<} \varepsilon\}$, which is called a non-Newtonian open ball of radius ε with center x. Similarly, one describes the non-Newtonian closed ball as $\bar{B}_\varepsilon^N(x) = \{y \in X : d_N(x,y) \dot{\leq} \varepsilon\}$.

Example 2.10: Consider the non-Newtonian metric space $(\mathbb{R}^+(N), d_N^*)$. From the definition of d_N^*, we can verify that the non-Newtonian open ball of radius $\varepsilon \dot{<} \dot{1}$ with center x_0 appears as $(x_0 \dot{-} \varepsilon, x_0 \dot{+} \varepsilon) \subset \mathbb{R}^+(N)$.

Definition 2.11 [3]: Let (X, d_N) be a non-Newtonian metric space and $A \subset X$.Then we call $x \in A$ a non-Newtonian interior point of A if there exists an $\varepsilon \dot{<} \dot{1}$ such that $B_\varepsilon^N(x) \subset A$. The collection of all interior points of A is called the non-Newtonian interior of A and is denoted by $int_N(A)$.

Definition 2.12 [3]: Let (X, d_N) be a non-Newtonian metric space and $A \subset X$. If every point of A is a non-Newtonian interior point of A, i.e., $A = int_N(A)$, then A is called a non-Newtonian open set.

Lemma 2.13 [3]: Let (X, d_N) be a non-Newtonian metric space. Each non-Newtonian open ball of X is a non-Newtonian open set.

Definition 2.14 [3]: Let (X, d_N) be a non-Newtonian metric space. A point $x \in X$ is said to be a non-Newtonian limit point of $S \subset X$ if and only if $(B_\varepsilon^N(x) \backslash \{x\}) \cap S \neq \emptyset$ for every $\varepsilon \dot{>} \dot{0}$. The set of all non-Newtonian limit points of the set S is denoted by $S_N^{'}$.

Definition 2.15 [4]: Let (X, d_N^N) and (Y, d_N^Y) be two non-Newtonian metric spaces and let $f: X \to Y$ be a function. If f satisfies the requirement that, for every $\varepsilon \dot{>} \dot{0}$, there exists $\delta \dot{>} \dot{0}$ such that $f(B_\delta^N(x)) \subset B_\varepsilon^N(f(x))$, then f is said to be non-Newtonian continuous at $x \in X$.

Example 2.16: Given a non-Newtonian metric space (X, d_N), define a non Newtonian metric on $X \times X$ by $p((x_1, x_2), (y_1, y_2)) = d_N(x_1, y_1) \dot{+} d_N(x_2, y_2)$. Then the non Newtonian metric $d_N : X \times X \to (\mathbb{R}^+(N), |\cdot|_N)$ is non Newtonian continuous on $X \times X$. To show this, let $(y_1, y_2), (x_1, x_2) \in X \times X$. Since we have $|d_N(y_1, y_2) \dot{-} d_N(x_1, x_2)|_N \ \dot{\leq}\ d_N(x_1, y_2) \dot{+} d_N(x_2, y_2)$, it is clear that d_N is non Newtonian continuous on $X \times X$. Now, we emphasize some properties of convergent sequences in a non Newtonian metric space.

Definition 2.17 [4]: A sequence (x_n) in a metric space $X = (X, d_N)$ is said to be convergent if for every given $\varepsilon \dot{>} \dot{0}$ there exist an $n_0 = n_0(\varepsilon) \in N and\ x \in X$ such that $d_N(x_n, x) \dot{<} \varepsilon$ for all $n > n_0$, and it is denoted by ${}^N\lim_{n\to\infty} x_n = x$ or $x_n \overset{N}{\to} x$, as $n \to \infty$.

Definition 2.18 [3]: A sequence (x_n) in a non-Newtonian metric space $X = (X, d_N)$ is said to be non-Newtonian Cauchy if for every $\varepsilon \dot{>} \dot{0}$ there exists an $n_0 = n_0(\varepsilon) \in N$ such that $d_N(x_n, x_m) \dot{<} \varepsilon$ for all $m, n > n_0$. Similarly, if for every non-Newtonian open ball $B_\varepsilon^N(x)$, there exists a natural number n_0 such that $n > n_0$, $x_n \in B_\varepsilon^N(x)$, then the sequence (x_n) is said to be non-Newtonian convergent to x.

The space X is said to be non-Newtonian complete if every non-Newtonian Cauchy sequence in X converges [4].

Proposition 2.19 [4]: Let $X = (X, d_N)$ be a non-Newtonian metric space. Then
(i) a convergent sequence in X is bounded and its limit is unique,
(ii) a convergent sequence in X is a Cauchy sequence in X.

Lemma 2.20 [3]: Let (X, d_N) be a non-Newtonian metric space, (x_n) a sequence in X and $x \in X$. Then $x_n \overset{N}{\to} x (n \to \infty)$ if and only if $d_N(x_n, x) \overset{N}{\to} \dot{0}\ (n \to \infty)$.

Lemma 2.21 [3]: Let (X, d_N) be a non-Newtonian metric space and let (x_n) be a sequence in X. If the sequence (x_n) is non-Newtonian convergent, then the non-Newtonian limit point is unique.

Theorem 2.22 [3]: Let (X, d_N^X) and (Y, d_N^Y) be two non-Newtonian metric spaces, $f: X \to Y$ a mapping and (x_n) any sequence in X. Then f is non-Newtonian continuous at the point $x \in X$ if and only if $f(x_n) \overset{N}{\to} f(x)$ for every sequence (x_n) with $x_n \overset{N}{\to} x$ $(n \to \infty)$.

Theorem 2.23 [3]: Let (X, d_N) be a non-Newtonian metric space and $S \subset X$. Then

(i) a point $x \in X$ belongs to $\bar{S}$ if and only if there exists a sequence (x_n) in S such that $x_n \overset{N}{\to} x$ $(n \to \infty)$,
(ii) the set S is non-Newtonian closed if and only if every non-Newtonian convergent sequence in S has a non-Newtonian limit point that belongs to S.

We now define the fixed point theorem on non-Newtonian metric spaces and give some examples.

Definition 2.24 [3]: Let X be a set and T a map from X to X. A fixed point of T is a point $x \in X$ such that $Tx = x$. In other words, a fixed point of T is a solution of the functional equation $Tx = x, x \in X$.

Definition 2.25 [3]: Suppose that (X, d_N) is a non-Newtonian complete metric space and $T: X \to X$ is any mapping. The mapping T is said to satisfy a non-Newtonian Lipchitz condition with $k \in \mathbb{R}(N)$ if $d_N(T(x), T(y)) \dot{\leq} k \dot{\times} d_N(x, y)$ holds for all $x, y \in X$.

If $k \dot{<} \dot{1}$, then T is called a non-Newtonian contraction mapping.

Theorem 2.26 [3]: Let T be a non-Newtonian contraction mapping on a non Newtonian complete metric space X. Then T has a unique fixed point.

Theorem 2.27 [3]: Let T be a mapping on a non-Newtonian complete metric space X into itself. Let T be a non-Newtonian contraction on a closed ball $\bar{B}_{\dot{r}}^N(x_0) = \{x \in X : d_N(x, x_0) \dot{\leq} \dot{r}\}$.

Suppose that $d_N(x_0, Tx_0) \dot{<} (\dot{1} \dot{-} k)\dot{r}$. Then the iterative sequence defined by $x_n = T^n x_0 = Tx_{n-1}$ converges to an $x \in \bar{B}_{\dot{r}}^N(x_0)$ and this x is the unique fixed point of T.

3. MAIN RESULTS

3.1. Commutative maps

Definition 3.1.1: Suppose that (X, d_N) is a non-Newtonian complete metric space and $S, T : X \to X$ be maps defined on X. Then S and T are said to be commutative if $S \circ T(x) = T \circ S(x)$ $\forall$ $x \in X$.

Proposition 3.1.2: Let T be a mapping of X into itself. Then T has a fixed point iff there is a constant map $S: X \to X$ which commutes with T.

Proof: By hypothesis there exists $a \in X$ and $S: X \to X$ such that $S(x) = a$ and $S(T(x)) = T(S(x))$ for all $x \in X$. We can therefore write $T(a) = T(S(a)) = S(T(a)) = a$, so that a fixed point of T.

The condition of necessity is proved along with theorem 3.1.4.

Lemma 3.1.3: Let $\{y_n\}$ be a sequence of complete non Newtonian metric space (X, d_N). If there exists $\alpha \in (0,1)$ such that $d_N(y_{n+1}, y_n) \dot{\leq} \alpha d_N(y_n, y_{n-1})$ for all n, then $\{y_n\}$ converges to a point in X.

Proof: Direct consequence of Theorem 2.26.

Theorem 3.1.4: Let T be a continuous mapping of a complete non Newtonian metric space (X, d_N) into itself. Then T has a fixed point in X if and only if there exists an $\alpha \in (\dot{0}, \dot{1})$ and a mapping $S: X \to X$ which commutes with T and satisfies

$$S(X) \subset T(X) \; and \; d_N(S(x), S(y)) \dot{\leq} \alpha d_N(T(x), T(y)) \; for \; all \; x, y \in X \tag{1}$$

Indeed T and S have a unique common fixed point if (1) holds.

Proof: To see that the condition stated is necessary, suppose that $T(a) = a$ for some $a \in X$. Define $S: X \to X$ by $S(x) = a$ for all $x \in X$. Then $S(T(x)) = a$ and $T(S(x)) = T(a) = a$ ($\forall x \in X$), so $S(T(x)) = T(S(x))$ for all $x \in X$ and T commutes with S. Moreover, $S(x) = a = T(a)$ for all $x \in X$ so that $S(x) \subset T(X)$. Finally, for any $\alpha \in (\dot{0}, \dot{1})$ we have for all $x, y \in X$,

$$d_N(S(x), S(y)) = d_N(a, a) = \dot{0} \dot{\leq} \alpha d_N(T(x), T(y)).$$

Thus, (1) holds.

On the other hand, suppose there is a map S of X into itself which commutes with T and for which (1) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point.

To this end, let $x_0 \in X$ and let x_1 be such that
$T(x_1) = S(x_0)$. In general, choose x_n so that
$$T(x_n) = S(x_{n-1}) \tag{2}$$

We can do this since $S(X) \subset T(X)$. The relation (1) and (2) imply that $d_N\big(T(x_{n+1}), T(x_n)\big) \dot{\leq} \alpha d_N\big(T(x_n), T(x_{n-1})\big)$ for all n. The lemma yields $t \in X$ such that
$$T(x_n) \overset{N}{\rightarrow} t \tag{3}$$

But then (2) implies that
$$S(x_n) \overset{N}{\rightarrow} t \tag{4}$$

Now since T is continuous, (1) implies that both S and T are continuous. Hence, (3) and (4) demand that $S\big(T(x_n)\big) \overset{N}{\rightarrow} S(t)$. But S and T commute so that $S\big(T(x_n)\big) = T\big(S(x_n)\big)$ for all n. Thus $S(t) = T(t)$, and consequently $T\big(T(t)\big) = T\big(S(t)\big) = S\big(S(t)\big)$ by commutativity. We can therefore infer
$$d_N\left(S(t), S\big(S(t)\big)\right) \dot{\leq} \alpha d_N\left(T(t), T\big(S(t)\big)\right) = \alpha d_N\left(S(t), S\big(S(t)\big)\right).$$

Hence, $d_N\left(S(t), S\big(S(t)\big)\right)(1-\alpha) \dot{\leq} 0$. Since $\alpha \in (0,1)$, $S(t) = S\big(S(t)\big)$. Since $\alpha \in (0,1)$, $S(t) = S\big(S(t)\big) = T\big(S(t)\big)$; i.e. $S(t)$ is common fixed point of T and S.

To see that T and S can have only one common fixed point, suppose that $x = T(x) = S(x)$ and $y = T(y) = S(y)$. Then (1) implies that $d_N(x,y) = d_N\big(S(x), S(y)\big) \dot{\leq} \alpha d_N\big(T(x), T(y)\big) = \alpha d_N(x,y)$, or $d_N(x,y)(1-\alpha) \dot{\leq} 0$. Since $\alpha \dot{<} 1, x = y$.

Corollary 3.1.5: Let T and S be commuting mappings of a complete non Newtonian metric space (X, d_N) into itself. Suppose that T is continuous and $S(X) \subset T(X)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that (i) $d_N\big(S^k(x), S^k(y)\big) \dot{\leq} \alpha d_N\big(T(x), T(y)\big)$ for all x and y in X, then T and S have a common fixed point.

Proof: Clearly, S^k commutes with T and $S^k(X) \subset S(X) \subset T(X)$. Thus the theorem pertains to S^k and T, so there exists a unique $a \in X$ such that $a = T(a) = S^k(a)$. But then, since T and S commute, we can write $S(a) = T\big(S(a)\big) = S^k\big(S(a)\big)$, which says that $S(a)$ is a c.f.p. of T and S^k. The uniqueness of a implies $a = S(a) = T(a)$.

Corollary 3.1.6: Let n be a positive integer and let K be a real number $\dot{>} 1$. If S is a continuous mapping of a complete non Newtonian metric space (X, d_N) onto itself such that $d_N\big(S^n(x), S^n(y)\big) \geq K d_N(x,y)$ for $x, y \in X$, then S has a unique fixed point.

Example 3.1.7: Let us consider an exponential calculus X, i.e., $\beta = exp, exp(x) = e^x \forall x \in \mathbb{R}$. Consider the maps $T, S: X \rightarrow X$ defined as, $T(x) = \{x \dot{/} \beta(2)\} \dot{+} \beta(3)$ and $S(x) = \{x \dot{/} \beta(3)\} \dot{+} \beta(4)$. Clearly, $S\big(T(x)\big) = \{x \dot{/} \beta(6)\} \dot{+} \beta(5) = T\big(S(x)\big)$. Also, $d_N\big(S(x), S(y)\big) \dot{\leq} \beta\left(\frac{2}{3}\right) \dot{\times} d_N\big(T(x), T(y)\big)$. So, these maps satisfy all the hypothesis of theorem 3.1.4. So, S and T has a common fixed point. The common fixed point of S and T is $x = \beta(6) = e^6$.

3.2. Weakly commutative maps

Definition 3.2.1: The self maps S and T of a non Newtonian metric space (X, d_N) are said to be weakly commutative iff $d_N\left(S\big(T(x)\big), T\big(S(x)\big)\right) \dot{\leq} d_N\big(S(x), T(x)\big)$ for all $x \in X$.

Remark 3.2.2: Every pair of commutative maps is weakly commutative but the converse is not true. Moreover, the weakly commutative maps commute on the coincidences points.

If $t \in X$, be the coincidence point of commutative maps S and T, i.e., $S(t) = T(t)$. Then, $d_N\left(S\big(T(t)\big), T\big(S(t)\big)\right) \dot{\leq} d_N\big(S(t), T(t)\big) \Rightarrow d_N\left(S\big(T(t)\big), T\big(S(t)\big)\right) \dot{\leq} d_N\big(S(t), S(t)\big) \Rightarrow S\big(T(t)\big) = T\big(S(t)\big)$.

Theorem 3.2.3: Let T be a mapping of a complete non Newtonian metric space (X, d_N) into itself. Then T has a fixed point in X if and only if there exists an $\alpha \in (\dot{0}, \dot{1})$ and a mapping $S: X \to X$ which commutes weakly with T and satisfies

$$S(X) \subset T(X) \text{ and } d_N(S(x), S(y)) \dot{\leq} \alpha d_N(T(x), T(y)) \text{ for all } x, y \in X \tag{5}$$

Indeed T and S have a unique common fixed point if (5) holds.

Proof: To see that the condition stated is necessary, suppose that $T(a) = a$ for some $a \in X$. Define $S: X \to X$ by $S(x) = a$ for all $x \in X$. Then $S(T(x)) = a$ and $T(S(x)) = T(a) = a$ $(\forall x \in X)$, so $S(T(x)) = T(S(x))$ for all $x \in X$ and T commutes with S. So, S and T are weakly commutative. Moreover, $S(x) = a = T(a)$ for all $x \in X$ so that $S(x) \subset T(X)$. Finally, for any $\alpha \in (\dot{0}, \dot{1})$ we have for all $x, y \in X$,

$$d_N(S(x), S(y)) = d_N(a, a) = \dot{0} \dot{\leq} \alpha d_N(T(x), T(y)).$$

Thus, (5) holds.

On the other hand, suppose there is a map S of X into itself which commutes weakly with T and for which (5) holds.

We show that the condition is sufficient to ensure that T and S have a unique fixed point.

To this end, let $x_0 \in X$ and let x_1 be such that $T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(x_n) = S(x_{n-1}) \tag{6}$$

We can do this since $S(X) \subset T(X)$. The relation (5) and (6) imply that $d_N(T(x_{n+1}), T(x_n)) \leq \alpha d_N(T(x_n), T(x_{n-1}))$ for all n. The lemma yields $t \in X$ such that

$$T(x_n) \overset{N}{\to} t \tag{7}$$

But then (6) implies that

$$S(x_n) \overset{N}{\to} t \tag{8}$$

Now, from the definition of weakly commutative

$$d_N\left(T(S(x_n)), S(T(x_n))\right) \dot{\leq} d_N(T(x_n), S(x_n))$$

$$\Rightarrow d_N(T(t), S(t)) \dot{\leq} d_N(t, t) \Rightarrow T(t) = S(t)$$

So, t is a coincidence point of S and T. So, $T(S(t)) = S(T(t)) = S(S(t))$. We can therefore infer

$$d_N\left(S(t), S(S(t))\right) \dot{\leq} \alpha d_N\left(T(t), T(S(t))\right) = \alpha d_N\left(S(t), S(S(t))\right).$$

Hence, $d_N\left(S(t), S(S(t))\right)(1 - \alpha) \dot{\leq} 0$. Since $\alpha \in (0,1)$, $S(t) = S(S(t)) = T(S(t))$, i.e. $S(t)$ is common fixed point of S and T.

To see that S and T can have only one common fixed point, suppose that $x = T(x) = S(x)$ and $y = T(y) = S(y)$.

Then (5) implies that $d_N(x, y) = d_N(S(x), S(y)) \dot{\leq} \alpha d_N(T(x), T(y)) = \alpha d_N(x, y)$, or $d_N(x, y)(1 - \alpha) \dot{\leq} 0$. Since $\alpha \dot{<} 1, x = y$.

Corollary 3.2.4: Let T and S be weakly commuting mappings of a complete non Newtonian metric space (X, d_N) into itself. Suppose, $S(X) \subset T(X)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that (i) $d_N(S^k(x), S^k(y)) \dot{\leq} \alpha d_N(T(x), T(y))$ for all x and y in X, then T and S have a unique common fixed point.

Proof: Clearly, S^k commutes weakly with T and $S^k(X) \subset S(X) \subset T(X)$. Thus the theorem pertains to S^k and T, so there exists a unique $a \in X$ such that $a = T(a) = S^k(a)$. But then, since T and S commute weakly, we can write $S(a) = T(S(a)) = S^k(S(a))$, which says that $S(a)$ is a c.f.p. of T and S^k. The uniqueness of a implies $a = S(a) = T(a)$.

Corollary 3.2.5: Let n be a positive integer and let K be a real number $\dot{>} 1$. If S is a self map of a complete non Newtonian metric space (X, d_N) such that $d_N(S^n(x), S^n(y)) \dot{\geq} K d_N(x, y)$ for $x, y \in X$, then S has a unique fixed point.

3.3. Compatible Maps

Definition 3.3.1: The self maps S and T of a non Newtonian metric space are said to be compatible iff $\lim_n d_N\left(S\big(T(x_n)\big), T\big(S(x_n)\big)\right) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n S(x_n) = \lim_n T(x_n) = t$ for some t in X.

Remark 3.3.2: Now, maps which are commutative are clearly compatible but the converse is not true.

Lemma 3.3.3: Let $S, T: (X, d_N) \to (X, d_N)$ be continuous, and let $F = \{x \in X | S(x) = T(x) = x\}$. Then S and T are compatible if any one of the following conditions is satisfied:

(a) if $S(x_n), T(x_n) \overset{N}{\to} t(\in X)$, then $t \in F$.

(b) $d_N\big(S(x_n), T(x_n)\big) \to 0$ implies $D(S(x_n), F) \overset{N}{\to} 0$.

where $D(x, F) = \max_{y\in F}\{d_N(x, y)\}$

Proof: Suppose that $\lim_n S(x_n) = \lim_n T(x_n) = t$ for some $t \in X$ (10)

If (a) holds, $S(t) = T(t) = t$. Then, the continuity of S and T on F imply $S\big(T(x_n)\big) \overset{N}{\to} S(t) = t$ and $T\big(S(x_n)\big) \overset{N}{\to} T(t) = t$, so that, $d_N\left(S\big(T(x_n)\big), T\big(S(x_n)\big)\right) \overset{N}{\to} 0$ as desired.

If (b) holds, the compatibility of S and T follow easily from (a) and (10) upon noting that F is closed since S and T are continuous.

Corollary 3.3.4: Suppose that S and T are continuous self maps of $\mathbb{R}$ such that $S - T$ is strictly increasing. If S and T have a common fixed point, then S and T are compatible.

Proof: Immediate, as $S(x_n), T(x_n) \overset{N}{\to} t$ implies that $F = \{t\}$.

Lemma 3.3.5: Let, $S, T: (X, d_N) \to (X, d_N)$ be compatible.

(1) If $S(t) = T(t)$, then $S\big(T(t)\big) = T\big(S(t)\big)$.

(2) Suppose that $\lim_n S(x_n) = \lim_n T(x_n) = t$, for some $t \in X$.

(a) If S is continuous at t, $\lim_n T\big(S(x_n)\big) = S(t)$.

(b) If S and T are continuous at t, then, $S(t) = T(t)$ and $S\big(T(t)\big) = T\big(S(t)\big)$.

Proof: Suppose that $S(t) = T(t)$, and let $x_n = t$ for n in $\mathbb{N}$. Then, $S(x_n), T(x_n) \overset{N}{\to} S(t)$, so that, $d_N\left(S\big(T(t)\big), T\big(S(t)\big)\right) = d_N\left(S\big(T(x_n)\big), T\big(S(x_n)\big)\right) \overset{N}{\to} 0$ by compatibility. Consequently, $d_N\left(S\big(T(t)\big), T\big(S(t)\big)\right) = 0$ and $S\big(T(t)\big) = T\big(S(t)\big)$, proving (1).

To prove (2)(a), note that if $T(x_n) \overset{N}{\to} t$, $S\big(T(x_n)\big) \overset{N}{\to} S(t)$ by the continuity of S. But if $S(x_n) \overset{N}{\to} t$, also since $d_N\left(T\big(S(x_n)\big), S(t)\right) \leq d_N\left(T\big(S(x_n)\big), S\big(T(x_n)\big)\right) + d_N\left(S\big(T(x_n)\big), S(t)\right)$ the compatibility of S and T require that, $d_N\left(T\big(S(x_n)\big), S(t)\right) \overset{N}{\to} 0$; i.e., $T\big(S(x_n)\big) \overset{N}{\to} S(t)$.

We prove (2)(b) by noting that $T\big(S(x_n)\big) \overset{N}{\to} S(t)$ by the continuity of S, whereas, $T\big(S(x_n)\big) \overset{N}{\to} T(t)$ by the continuity of T. Thus, $S(t) = T(t)$ by the uniqueness of limit, and therefore $T\big(S(x_n)\big) = S\big(T(x_n)\big)$ by part (1).

Theorem 3.3.6: Let $S\ and\ T$ be non Newtonian continuous compatible maps of a complete non Newtonian metric space (X, d_N) into itself. Then $S\ and\ T$ have a unique common fixed point in X if there exists an $\alpha \in \big(\dot{0}, \dot{1}\big)$ and they satisfy

$$S(X) \subset T(X)\ and\ d_N\big(S(x), S(y)\big) \dot{\leq} \alpha d_N\big(T(x), T(y)\big)\ for\ all\ x, y \in X \qquad (11)$$

Proof: Suppose there is a map S of X into itself which is compatible with T and for which (11) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point.

To this end, let $x_0 \in X$ and let x_1 be such that $T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(x_n) = S(x_{n-1}) \qquad (12)$$

We can do this since $S(X) \subset T(X)$. The relation (11) and (12) imply that $d_N\big(T(x_{n+1}), T(x_n)\big) \dot{\leq} \alpha d_N\big(T(x_n), T(x_{n-1})\big)$ for all n. The lemma yields $t \in X$ such that

$$T(x_n) \xrightarrow{N} t \tag{13}$$

But then (12) implies that

$$S(x_n) \xrightarrow{N} t \tag{14}$$

Now, from the definition of compatibility,

$$\lim_n d_N \left(S\big(T(x_n)\big), T\big(S(x_n)\big)\right) = \dot{0}$$

Now, by continuity of S *and* T and the lemma, $S(t) = T(t)$ and $S\big(T(t)\big) = T\big(S(t)\big)$. We can therefore infer

$$d_N\left(S(t), S\big(S(t)\big)\right) \dot{\leq} \alpha d_N\left(T(t), T\big(S(t)\big)\right) = \alpha d_N\left(S(t), S\big(T(t)\big)\right) = \alpha d_N\left(S(t), S\big(S(t)\big)\right).$$

Hence, $d_N\left(S(t), S\big(S(t)\big)\right)(1-\alpha) \dot{\leq} \dot{0}$. Since $\alpha \in (\dot{0}, \dot{1})$, $S(t) = S\big(S(t)\big) = T\big(S(t)\big)$, i.e. $S(t)$ is common fixed point of S and T.

To see that S and T can have only one common fixed point, suppose that $x = T(x) = S(x)$ and $y = T(y) = S(y)$. Then (11) implies that $d_N(x,y) = d_N\big(S(x), S(y)\big) \dot{\leq} \alpha d_N\big(T(x), T(y)\big) = \alpha d_N(x,y)$, or $d_N(x,y)(1-\alpha) \dot{\leq} \dot{0}$. Since, $\alpha \dot{<} \dot{1}$, so we infer that $x = y$.

Corollary 3.3.7: Let T and S be continuous compatible maps of a complete non Newtonian metric space (X, d_N) into itself. Suppose, $S(X) \subset T(X)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that (i) $d_N\big(S^k(x), S^k(y)\big) \leq \alpha d_N\big(T(x), T(y)\big)$ for all x and y in X, then T and S have a common fixed point.

Proof: Clearly, S^k is compatible with T and $S^k(X) \subset S(X) \subset T(X)$. Thus, the theorem pertains to S^k and T, so there exists a unique $a \in X$ such that $a = T(a) = S^k(a)$. But then, since T and S are compatible, we can write $S(a) = T\big(S(a)\big) = S^k\big(S(a)\big)$, which says that $S(a)$ is a c.f.p. of T and S^k. The uniqueness of a implies $a = S(a) = T(a)$.

Example 3.3.8: Let us consider an exponential calculus X, i.e., $\beta = exp, exp(x) = e^x \forall x \in \mathbb{R}$. Consider the maps, $S, T: X \to X$ defined as $S(x) = \beta(2) \dot{-} x^{2_N}$ and $T(x) = x^{2_N}$. Now, $|S(x_n) - T(x_n)|_N = \beta(2) \dot{\times} \left|\dot{1} \dot{-} x^{2_N}\right|_N \xrightarrow{N} \dot{0}$ iff $x_n = \pm\dot{1}$ and $\lim_n S(x_n) = \lim_n T(x_n) = \dot{1}$. So, these are compatible but clearly not weakly commutative, e.g. $x = \dot{0}$. Now, $d_N\big(S(x), S(y)\big) \dot{\leq} \beta\left(\frac{1}{2}\right) \dot{\times} d_N\big(T(x), T(y)\big)$. Hence, S and T satisfy all the hypotheses of the theorem 3.3.6 and hence have a unique common fixed point. The fixed point of these two mappings is $x = \dot{1} = e$.

3.4. Weakly Compatible Maps:

Definition 3.4.1: Two maps S and T are said to be weakly compatible if they commute at coincidence points.

Remark 3.4.2: Commutative ⇒ weakly commutative ⇒ compatible ⇒ weakly compatible. But the converse of the above implications does not hold.

Theorem 3.4.3: Let S *and* T be weakly compatible maps of a complete non Newtonian metric space (X, d_N) into itself. Then S *and* T have a unique common fixed point in X if there exists an $\alpha \in (\dot{0}, \dot{1})$ and they satisfy the following two conditions:

$$S(X) \subset T(X) \text{ and } d_N\big(S(x), S(y)\big) \dot{\leq} \alpha d_N\big(T(x), T(y)\big) \text{ for all } x, y \in X \tag{15}$$

Any one of the subspace $S(X)$ or $T(X)$ is complete (16)

Proof: Suppose, there is a map S of X into itself which is weakly compatible with T and for which (15) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point.

To this end, let $x_0 \in X$ and let x_1 be such that

$T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(x_n) = S(x_{n-1}) \tag{17}$$

We can do this since $S(X) \subset T(X)$. The relation (15) and (16) imply that $d_N\big(T(x_{n+1}), T(x_n)\big) \dot{\leq} \alpha d_N\big(T(x_n), T(x_{n-1})\big)$ for all n. The lemma yields $t \in X$ such that

$$T(x_n) \xrightarrow{N} t \tag{18}$$

But then (17) implies that

$$S(x_n) \xrightarrow{N} t \tag{19}$$

Since, either $S(X)$ or $T(X)$ is complete, for certainty assume that $T(X)$ is complete subspace of X then the subsequence of $\{x_n\}$ must get a limit in $T(X)$. Call it be t. Let $u \in T^{-1}(t)$. Then $T(u) = t$ as $\{x_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{x_n\}$ also convergent implying thereby the convergence of subsequence of the convergent sequence. Now, we show that $S(u) = t$.

On setting $x = x_n, y = u$ in (15), we have

$$d_N\big(S(x_n), S(u)\big) \dot{\leq} \alpha d_N\big(T(x_n), T(u)\big)$$

Proceeding limit as $n \to \infty$, we get $d_N\big(t, S(u)\big) \dot{\leq} \alpha d_N\big(t, T(u)\big) \Rightarrow d_N\big(t, S(u)\big) \dot{\leq} \dot{0} \Rightarrow S(u) = t$. Thus, u is a coincidence point of S and T. Since S and T are weakly compatible, it follows that $S\big(T(u)\big) = T\big(S(u)\big)$.

We can therefore infer

$$d_N\Big(S(u), S\big(S(u)\big)\Big) \dot{\leq} \alpha d_N\Big(T(u), T\big(S(u)\big)\Big) = \alpha d_N\Big(S(u), S\big(T(u)\big)\Big) = \alpha d_N\Big(S(u), S\big(S(u)\big)\Big).$$

Hence, $d_N\Big(S(u), S\big(S(u)\big)\Big)(1-\alpha) \dot{\leq} \dot{0}$. Since $\alpha \in (0,1)$, $S(u) = S\big(S(u)\big) = T\big(S(u)\big)$, i.e. $S(u)$ is common fixed point of S and T.

To see that S and T can have only one common fixed point, suppose that $x = T(x) = S(x)$ and $y = T(y) = S(y)$. Then (11) implies that $d_N(x,y) = d_N\big(S(x), S(y)\big) \leq \alpha d_N\big(T(x), T(y)\big) = \alpha d_N(x,y)$, or $d_N(x,y)(1-\alpha) \leq \dot{0}$. Since, $\alpha < 1, x = y$.

Corollary 3.4.4: Let T and S be weakly compatible maps of a complete non Newtonian metric space (X, d_N) into itself. Suppose, $S(X) \subset T(X)$. If there exists $\alpha \in \big(\dot{0}, \dot{1}\big)$ and a positive integer k such that (i) $d_N\big(S^k(x), S^k(y)\big) \dot{\leq} \alpha d_N\big(T(x), T(y)\big)$ for all x and y in X, (ii) either $S^k(X)$ or $T(X)$ is complete subspace in X, then T and S have a common fixed point.

Proof: Clearly, S^k is compatible with T and $S^k(X) \subset S(X) \subset T(X)$. Thus, the theorem pertains to S^k and T, so there exists a unique $a \in X$ such that $a = T(a) = S^k(a)$. But then, since T and S are compatible, we can write $S(a) = T\big(S(a)\big) = S^k\big(S(a)\big)$, which says that $S(a)$ is a c.f.p. of T and S^k. The uniqueness of a implies $a = S(a) = T(a)$.

REFERENCES

1. Aamri, M, Moutawakil, DEl, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 2002, 270, 181-188.
2. Banach, S, Sur les operations dans les ensembles abstraits et leurs applications aux equations integrales, Fund. Math. 1922, 3, 133–181.
3. Binba¸ sıoˇglu, D, Demiriz, S, Tu¨rkogˇlu, D, Fixed points of non-Newtonian contraction mappings on non-Newtonian metric spaces, J. Fixed Point Theory Appl. 2016, 18, 213–224.
4. C¸akmak, AF, Ba¸sar, F, Some new results on sequence spaces with respect to non-Newtonian calculus, J. Inequal. Appl., 2012, 2012, 17 pages.
5. Choudhury, BS, Das, KP, A new contraction principle in Menger spaces, Acta Mathematica Sinicia, English Series 2008, 24(8), 1379-1386.
6. Choudhury, BS, Kumar, S, Asha, Das, K, Some fixed point theorems in G-metric spaces, Math. Sci. Lett. 2012, 1(1), 25-31.
7. Das, KM, Naik, KV, Common fixed-point theorems for commuting maps on a metric space, Proc. Amer. Math. Soc. 1979, 77, 369-373.
8. Grossman, M, Katz, R, Non-Newtonian Calculus, Lowell Technological Institute, 1972.
9. Jungck, G, Commuting mappings and fixed point, Amer. Math. Monthly 1976, 83, 261-263.
10. Jungck, G, Compatible mappings and common fixed points, International J. Math. Math. Sci. 1986, 9(4), 771-779.
11. Jungck, G, Common fixed points for non-continuous non-self maps on non-metric spaces, Far East J. Mathe. Sci. 1996, 4 (2), 199-212.
12. Jungck, G, Murthy, PP, Cho, YJ, Compatible mappings of type (A) and common fixed points, Math. Japon. 1993, 38, 381 – 390.
13. Kannan, R, Some results on fixed point, Bull. Cal. Mathe. Soc. 1968, 60, 71-76.
14. Kannan, R, Some results on fixed point II, Amer. Mathe. Monthly 1969, 76, 405-408.
15. Rani, A, Kumar, S, Common fixed point theorems in generalized menger spaces with t-norm t of hadžić-type, Adv. Fixed Point Theory 2013, 3(1), 151-163.

16. Rani, A, Kumar, S, Kumar, N, Garg, SK, Common Fixed Point Theorems for Compatible and Weakly Compatible Maps in G-Metic Spaces, Applied Math. 2012, 3, 1128-1134.
17. Sessa, S, On a weak commutativity condition of mappings in fixed point considerations, Publications DeL "Institut Mathematique", Nouvelle Serie Tome 1982, 32, 149-153.
18. Z-Dos`enovic`, T, A common fixed point theorem for compatible mappings in fuzzy metric spaces using implicit relation, Acta Math. Hang. 2009, 125(4), 357-368.

Source of support: Nil, Conflict of interest: None Declared