

COMMON FIXED POINTS
IN COMPLEX-VALUED b-METRIC SPACES SATISFYING A SET OF RATIONAL NEQUALITIES

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ABSTRACT

In 1989, Bakhtin introduced the notion of b-metric space (I. A. Bakhtin, The contraction principal in quasi-metric spaces, *Functional Analysis* 30(1989), 26-37) as a generalization of metric space in which the triangle inequality was relaxed. Further, in 1993, Czerwik first proved a contraction mapping theorem for this space (S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis* 1(1993), 5-11) which generalized the well known Banach contraction mapping principal. In 2011, Azam et al. introduced the notion of complex-valued metric space (A. Azam, B. Fisher and M. S. Khan, Common fixed point theorems in complex-valued metric spaces, *Numerical Functional Analysis & Optimization* 32(3)(2011), 243-253) to obtained a common fixed point result for a pair of self-mappings satisfying a rational inequality. Meanwhile, Jungck relaxed the concept of commutativity of a pair of mappings by compatibility [9], and further by weakly compatibility [10]. In this paper, we will prove some common fixed point theorems in complex-valued b-metric spaces for two pairs of self-mappings satisfying a set of rational inequalities using the weakly compatible mappings. Our result generalizes many results in the existing literature.

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Keywords and Phrases: Banach contraction mapping principal, common fixed point, complete metric space, complex-valued metric space, complex-valued b-metric space, weakly compatible mappings.

Short Title: Common fixed points in complex-valued b-metric spaces

The first page will contain the title, the authors, an abstract, subject Classification same as the AMS, a short title, a list of key words and phrases and the complete address (es) of the author(s) with e-mail.

The novelty embodied in my work, or in the approach taken in my research: In this paper, I have unified the notions of complex-valued metric space as well as of b-metric space. Since, every b-metric space is a metric spaces, my result will automatically apply for ordinary metric space (X, d) . Also, since, a complex-valued metric d is a function from a set $X \times X$ into \mathbb{C} , our theorem generalizes many quasi-contraction mapping on real-valued metric. This theorem also relaxes the commutativity and compatibility of mapping-pair, as well. Further, our theorem generalizes well known current results of complex-valued metric space like [1, 3, 4, 7, 11, 12, 16, 17, 19, 20, 21, 22, 25], results of b-metric spaces of [8, 15], and theorems on complex-valued b-metric spaces in the existing literature, like [14, 18].

1. INTRODUCTION

Banach contraction principal [5] is a basic result in fixed point theory. This theorem has been generalized in many ways. Bakhtin [6] introduced the notion of b-metric space as a generalization of metric space in which the triangle inequality is relaxed. Czerwik [8] proved a contraction theorem in b-metric space which generalized the Banach contraction principal. Malhotra and Bansal [15] proved some common coupled fixed point theorems for generalized contraction in b-metric spaces. Azam et al. [2] introduced the notion of complex valued metric space as a generalization of metric space. They established sufficient conditions for the existence of common fixed points of a pair of mappings in this space satisfying a rational inequality. Jungck [9] introduced the notion of compatible mappings for a pair of mappings. This notion was further generalized to weakly compatible mappings [10]. In this line, Shukla-Pagey [20] and Verma-Pathak [24] proved some common fixed point theorems in complex valued metric spaces. More results on complex-valued metric spaces can be found in [1], [2], [3], [4], [7], [11], [12], [13], [16], [17], [19], [20], [21], [22], [23], [24], [25] etc. In 2014, Mukheimer [14] and Rao et al. [18] proved common fixed point theorems using complex valued b-metric space. In this paper, we prove a common fixed point theorem for two pairs of weakly compatible mappings in a complete complex valued b-metric space satisfying a set of contraction condition. Our theorem generalizes many results in the literature.

2. PRELIMINARIES

Let C be the set of complex numbers $z=a+ib$, where a, b are real numbers, a is called $Re(z)$ and b is called $Im(z)$. A complex valued metric d is a function from a set $X \times X$ into C . Let $z_1, z_2 \in C$. Define a partial order \ll on C as follows: $z_1 \ll z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. It follows that $z_1 \ll z_2$ if one of the following conditions satisfies:

- (i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,
- (ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,
- (iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,
- (iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In (i), (ii) and (iii), we have $|z_1| < |z_2|$. In (iv), we have $|z_1| = |z_2|$. So, $|z_1| \leq |z_2|$ whenever $z_1 \ll z_2$. We will write $z_1 \prec z_2$ if only (iii) satisfy. Hence $z_1 \prec z_2$ implies $|z_1| < |z_2|$.

Remark ([11]): We note that the following statements hold. Let $z_1, z_2, z_3 \in C$.

- (i) $a, b \in R$ and $a \leq b$ implies $az \ll bz$,
- (ii) $0 \ll z_1 \not\ll z_2$ implies $|z_1| < |z_2|$,
- (iii) $z_1 \ll z_2$ and $z_2 \prec z_3$ implies $z_1 \prec z_3$.

Azam *et al.* [2] defined complex-valued metric space (X, d) in the following way:

Definition 2.1 ([2]): Let X is a nonempty set. Suppose that the mapping $d: X \times X \rightarrow C$ satisfies the following conditions:

- (A1) $0 \ll d(x, y)$, and $d(x, y) = 0$ if and only if $x=y$, for all $x, y \in X$;
- (A2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
- (A3) $d(x, y) \ll d(x, z) + d(z, y)$, for all $x, y, z \in X$;

then d is called complex-valued metric, and (X, d) is called a complex-valued metric space.

Definition 2.2 ([15]): Let X is a nonempty set and let $s \geq 1$ be a real number. The mapping $d: X \times X \rightarrow R$ is called b-metric space if following three conditions satisfy:

- (B1) $0 \leq d(x, y)$, and $d(x, y) = 0$ if and only if $x=y$, for all $x, y \in X$,
- (B2) $d(x, y) = d(y, x)$, for all $x, y \in X$,
- (B3) $d(x, y) \leq s[d(x, z)+d(z, y)]$, for all $x, y, z \in X$. The number $s \geq 1$ is called the coefficient of b-metric space.

Example 2.3: Let $X = \{-1, 0, 1\}$. Define mapping $d: X \times X \rightarrow R^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X, d(x, x) = 0, x \in X$ and $d(-1, 0) = 4, d(-1, 1) = 1, d(0, 1) = 2$. Then (X, d) is a b-metric space, but not a metric space, since the triangle inequality is not satisfied. Indeed, $d(-1, 0)+d(0, 1)=4 + 2 = 6 > 1 = d(-1, 1)$ and $d(-1, 1)+d(-1, 0) = 1+4 = 5 > 2 = d(1, 0)$ are both true; but $d(-1, 1)+d(1, 0) = 1+2 = 3 < 4 = d(-1, 0)$ is not true. So, (X, d) is b-metric space with $s = 4/3$.

Example 2.4: Let $X = [0, 1]$ and $d: X \times X \rightarrow [0, \infty)$ be defined by $d(x, y) = (x - y)^2$, for all $x, y \in X$. Clearly, (X, d) is a b-metric space.

Example 2.5 ([15]): Let (X, d) be a metric space, $d^*(x, y) = (d(x, y))^p$ with $p > 1$, then (X, d^*) is a b-metric space with $s=2^{p-1}$.

Remark 2.6 ([13]): In the b-metric space (X, d) , the following assertions hold:

- (i) A convergent sequence has a unique limit.
- (ii) Each convergent sequence is Cauchy sequence.
- (iii) In general, a b-metric is not continuous (See Example in [13]).

On generalizing (A3), *i.e.*, triangle inequality, the complex valued b-metric space is defined in the following way:

Definition 2.7 ([14], [15], [18]): Let X be a nonempty set and mapping $d: X \times X \rightarrow C$ satisfy the following conditions:

- (C1) $0 \ll d(x, y)$, and $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$,
- (C2) $d(x, y) = d(y, x)$, for all $x, y \in X$,
- (C3) $d(x, y) \ll s[d(x, z)+d(z, y)]$, for all $x, y, z \in X$, where $s \geq 1$ is a real number, then d is called complex valued b-metric in X , and (X, d) is called complex valued b-metric space.

Example 2.8 ([18]): Let $X = [0, 1]$. Define a complex-valued metric $d: X \times X \rightarrow C$ by: $d(x, y) = |x-y|^2 + i|x-y|^2$, for all $x, y \in X$, then (X, d) is a complex valued b-metric space with $s=2$.

Remark 2.9: If $s=1$, then the complex-valued b-metric space always reduces to a complex-valued metric space. Thus every complex-valued metric space is a complex valued b-metric space, but not conversely. This generalizes the notion of a complex valued b-metric space over complex-valued metric space.

Definitions of interior point, limit point, open and closed sets, Hausdorff topology on X, the convergent sequence, Cauchy sequence and complete complex-valued metric space, can be seen from Rao *et.al.* [18]. Following lemmas are useful in our paper. Let us define weakly compatibility in the complex-valued b-metric space, below:

Lemma 2.10 ([18]): Let $\{x_n\}_{n \geq 1}$ be a sequence of a complex-valued b-metric space (X, d) , then $\{x_n\}_{n \geq 1}$ converges to x if and only if $\lim_{n \rightarrow \infty} |d(x_n, x)| = 0$.

Lemma 2.11 ([18]): Let $\{x_n\}_{n \geq 1}$ be a sequence of a complex-valued b-metric space (X, d) , then $\{x_n\}_{n \geq 1}$ is a Cauchy sequence if and only if $\lim_{n \rightarrow \infty} |d(x_n, x_{n+m})| = 0$, where $m \in \mathbb{N}$.

Definition 2.12 ([10]): Let (X, d) be a complex-valued metric space. A pair of self-mappings $A, S: X \rightarrow X$ is called weakly compatible if they commute at their coincidence points. That is, $ASu = SAu$, whenever $Au = Su$, for some $u \in X$.

Example 2.13: Let $X = \mathbb{R}$. Define complex valued b-metric $d: X \times X \rightarrow \mathbb{C}$ by $d(x, y) = |x-y|^2 + i|x-y|^2$, for all $x, y \in X$. Define mappings $f, g: X \rightarrow X$, by $f(x) = x/4, g(x) = x/5$, for all $x \in X$. Then f and g have coincidence point at $x = 0$. Now at this point, $fg(0) = gf(0)$. Thus (f, g) is weakly compatible at $x=0$.

In 2011, Azam *et al.* ([2]) proved the following result in complex valued metric space:

Theorem 2.14 ([2]): Let (X, d) be a complete complex-valued metric space and let λ, μ be non-negative real numbers such that $\lambda + \mu < 1$. Suppose that $S, T: X \rightarrow X$ are mappings satisfying

$$d(Sx, Ty) \ll \lambda \cdot d(x, y) + \mu \cdot d(x, Sx)d(y, Ty) / \{1 + d(x, y)\} \tag{2.1}$$

for all $x, y \in X$. Then S and T have a unique common fixed point in X .

This result was generalized by Mukheimer ([14]) for a complex valued b-metric space in the following way:

Theorem 2.15 ([14]): Let (X, d) be a complete complex-valued b-metric space with parameter $s \geq 1$. Suppose that $S, T: X \rightarrow X$ are mappings satisfying:

$$d(Sx, Ty) \ll \lambda \cdot d(x, y) + \{\mu \cdot d(x, Sx)d(y, Ty) / [1 + d(x, y)]\}, \text{ for all } x, y \in X \tag{2.2}$$

where λ, μ are non-negative real numbers with $s\lambda + \mu < 1$. Then S and T have a unique common fixed point in X .

Mukheimer [14] also proved the following result, to generalize the main theorem of Bhatt *et al.* [3]:

Theorem 2.16 ([14]): Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and let $S, T: X \rightarrow X$ be mappings satisfying:

$$d(Sx, Ty) \ll \{a[d(x, Sx)d(x, Ty) + d(y, Ty)d(y, Sx)] / [d(x, Ty) + d(y, Sx)]\} \tag{2.3}$$

for all $x, y \in X$, where $sa \in [0, 1)$. Then S and T have a unique common fixed point in X .

Other results in this line can be obtained in Ahmad *et.al.* [1] and Bhatt *et.al.* [3] for complex-valued metric space, and Mukheimer [14] for complex-valued b-metric space. In this paper, we prove some common fixed point theorems for two pairs of weakly compatible mappings satisfying a set of rational inequalities in complex valued b-metric space.

3. MAIN RESULTS

Theorem 3.1: Let $A, B, S, T: X \rightarrow X$ be self-mappings of a complete complex valued b-metric space (X, d) , with metric parameter $s \geq 1$. Suppose that the set $B(X)$ is closed. If the following conditions satisfy:

(i) $A(X) \text{ subseteq } T(X), B(X) \text{ subseteq } S(X)$,

(ii) Let the set of inequalities is:

$$d(Ax, By) \ll q \ddot{U}_{x,y}(A, B, S, T), \text{ for all } x, y \in X \tag{3.1}$$

where q is a non-negative real number such that $0 \leq q < 1/(s^2 + s)$, and

$$\begin{aligned} \ddot{U}_{x,y}(A, B, S, T) \in \{ & d(Sx, Ty), d(Ax, Sx)d(By, Ty) / [d(Sx, Ty) + d(Ax, By)], d(By, Sx)d(Ax, Ty) / [d(Sx, Ty) + d(Ax, By)], \\ & d(Ax, Sx)d(By, Sx) / [d(Ax, By) + d(Sx, Ty)], d(Ax, Ty)d(By, Ty) / [1 + d(Sx, Ty)], d(Ax, Sx)d(By, Ty) / [d(Sx, Ty) + d(Ax, By)], \\ & d(Sx, Ty)d(Ax, By) / [d(Sx, Ty) + d(Ax, By)], d(By, Sx)d(Sx, Ty) / [d(Ax, By) + d(Sx, Ty)], \\ & d(Ax, Sx)d(Sx, Ty) / [d(Ax, Sx) + d(Sx, Ty)], \frac{1}{2}s \cdot d(Sx, Ty)d(By, Ty) / [d(Ax, By) + d(Sx, Ty)] \}, \end{aligned}$$

where $\ddot{U}_{x,y}(A, B, S, T)$ denotes the "selection of one co-ordinate amongst these options"

(iii) the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have unique common fixed point in X .

Proof: Since $s \geq 1$ so that $1/(s^2 + s) \leq 1/2$, and $0 \leq q < 1/(s^2 + s)$, so, $0 \leq qs < 1/(s+1) \leq 1/2 < 1$, and $1 - qs > 1 - 1/(s+1)$. Also, we have $0 \leq q \leq 1/2$. Let x_0 be an arbitrary point of X . Define a sequence $\{x_n\}$ in X , such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n} \quad \text{and} \quad Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}, \text{ for all } n = 0, 1, 2, 3 \tag{3.2}$$

We will show that $\{y_n\}$ is a Cauchy sequence. If $x_{2n} \neq x_{2n+1}$ then putting $x = x_{2n}$ and $y = x_{2n+1}$ in condition (ii), and denoting $d_n = d(y_{n+1}, y_n)$, we have

$$d_{2n} = d(y_{2n+1}, y_{2n}) = d(Ax_{2n}, Bx_{2n+1}) \ll q. \ddot{U}x_{2n}, x_{2n+1} (A, B, S, T) \tag{3.3}$$

where $\ddot{U}x_{2n}, x_{2n+1} (A, B, S, T) \in \{d_{2n-1}, d_{2n-1}d_{2n}/[d_{2n-1}+d_{2n}], 0, d_{2n-1}d(y_{2n-1}, y_{2n+1})/[d_{2n}+d_{2n-1}], 0, d_{2n-1}d_{2n}/[d_{2n-1}+d_{2n}], d_{2n-1}d_{2n}/[d_{2n-1}+d_{2n}], d(y_{2n-1}, y_{2n+1})d_{2n-1}/[d_{2n}+d_{2n-1}], d_{2n-1}d_{2n-1}/[d_{2n-1}+d_{2n-1}], 1/2sd_{2n-1}d_{2n}/[d_{2n}+d_{2n-1}]\}$

i.e., $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) \in \{d_{2n-1}, d_{2n-1}d_{2n}/[d_{2n-1}+d_{2n}], 0, sd_{2n-1}[d_{2n-1}+d_{2n}]/[d_{2n}+d_{2n-1}], 0, d_{2n-1}d_{2n}/[d_{2n-1}+d_{2n}], d_{2n-1}d_{2n}/[d_{2n-1}+d_{2n}], s.[d_{2n-1}+d_{2n}].d_{2n-1}/[d_{2n}+d_{2n-1}], d_{2n-1}d_{2n-1}/2d_{2n-1}, 1/2sd_{2n-1}d_{2n}/[d_{2n}+d_{2n-1}]\}$ (using (C3))

We have following cases to consider:

- Case-1:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = d_{2n-1}$, then from (3.1), we have $d_{2n} \ll q.d_{2n-1}$ implies $|d_{2n}| \leq q|d_{2n-1}|$.
- Case-2:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = d_{2n-1}d_{2n}/(d_{2n-1}+d_{2n})$, then from (3.1), $d_{2n} \ll q.d_{2n-1}d_{2n}/(d_{2n-1}+d_{2n})$, i.e., $d_{2n} \ll -(1-q).d_{2n-1}$, whence $|d_{2n}| \leq (1-q)|d_{2n-1}|$.
- Case-3:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = 0$, then from (3.1), $d_{2n} \ll q.0$ implies $|d_{2n}| \leq 0$.
- Case-4:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = s.d_{2n-1}(d_{2n-1}+d_{2n})/(d_{2n}+d_{2n-1})$, then from (3.1), $d_{2n} \ll qs.d_{2n-1}$ implies $|d_{2n}| \leq qs|d_{2n-1}|$.
- Case-5:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = 0$, then from (3.1), $d_{2n} \ll q.0$ implies $|d_{2n}| \leq 0$.
- Case-6:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = d_{2n-1}d_{2n}/(d_{2n-1}+d_{2n})$, then from (3.1), $d_{2n} \ll q.d_{2n-1}d_{2n}/(d_{2n-1}+d_{2n})$ implies $d_{2n} \ll -(1-q)d_{2n-1}$ i.e., $|d_{2n}| \leq (1-q)|d_{2n-1}|$.
- Case-7:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = d_{2n-1}.d_{2n}/[d_{2n-1}+d_{2n}]$, then $d_{2n} \ll -(1-q)d_{2n-1}$, which implies $|d_{2n}| \leq (1-q).|d_{2n-1}|$.
- Case-8:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = s.[d_{2n-1}+d_{2n}].d_{2n-1}/[d_{2n}+d_{2n-1}]$, then $d_{2n} \ll qs.d_{2n-1}$, i.e., $|d_{2n}| \leq qs.|d_{2n-1}|$, where $qs < 1$.
- Case-9:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = q.d_{2n-1}d_{2n-1}/2d_{2n-1}$, then from (3.1), $d_{2n} \ll 1/2q.d_{2n-1}$, i.e., $|d_{2n}| \leq 1/2q.|d_{2n-1}|$
- Case-10:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = 1/2s.d_{2n-1}d_{2n}/[d_{2n}+d_{2n-1}]$, then from (3.1), $d_{2n} \ll 1/2qs.d_{2n-1}d_{2n}/[d_{2n}+d_{2n-1}]$, or, $d_{2n} \ll -(1-1/2qs).d_{2n-1}$, implies $|d_{2n}| \leq (1-1/2qs)|d_{2n-1}|$.

Hence in all cases, we obtain $|d_{2n}| \leq \lambda .|d_{2n-1}|$ (3.4),

where $\lambda = \max\{q, 1-q, 0, qs, 1/2q, 1-1/2qs\} < 1$, as $qs \leq 1/2$ and $0 \leq q < 1/(s^2+s) \leq 1/2$.

Similarly, by putting $x = x_{2n+2}$ and $y = x_{2n+1}$ in condition (ii), we have:

$$d(Ax_{2n+2}, Bx_{2n+1}) = d_{2n+1} \ll q \ddot{U}x_{2n+2}, x_{2n+1} (A, B, S, T), \text{ where } 0 < q \in R \text{ such that } 0 \leq q < 1/(s^2+s), \text{ and}$$

$\ddot{U}x, y(A, B, S, T) \in \{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})d(y_{2n+1}, y_{2n})/[d(y_{2n+1}, y_{2n})+d(y_{2n+2}, y_{2n+1})], d(y_{2n+1}, y_{2n+1})d(y_{2n+2}, y_{2n})/[d(y_{2n+1}, y_{2n})+d(y_{2n+2}, y_{2n+1})], d(y_{2n+2}, y_{2n+1})d(y_{2n+1}, y_{2n+1})/[d(y_{2n+2}, y_{2n+1})+d(y_{2n+1}, y_{2n})], d(y_{2n+2}, y_{2n}).d(y_{2n+1}, y_{2n})/[1+d(y_{2n+1}, y_{2n})], d(y_{2n+2}, y_{2n+1})d(y_{2n+1}, y_{2n})/[d(y_{2n+1}, y_{2n})+d(y_{2n+2}, y_{2n+1})], d(y_{2n+1}, y_{2n}).d(y_{2n+2}, y_{2n+1})/[d(y_{2n+1}, y_{2n})+d(y_{2n+2}, y_{2n+1})], d(y_{2n+1}, y_{2n+1})d(y_{2n+1}, y_{2n})/[d(y_{2n+2}, y_{2n+1})+d(y_{2n+1}, y_{2n})], d(y_{2n+2}, y_{2n+1})d(y_{2n+1}, y_{2n})/[d(y_{2n+2}, y_{2n+1})+d(y_{2n+1}, y_{2n})]\}$

$\ll \{d_{2n}, d_{2n+1}d_{2n}/(d_{2n}+d_{2n+1}), 0, 0, (d_{2n+1}+d_{2n}).d_{2n}/(1+d_{2n}), d_{2n+1}d_{2n}/(d_{2n}+d_{2n+1}), d_{2n}.d_{2n+1}/(d_{2n}+d_{2n+1}), 0, d_{2n+1}d_{2n}/(d_{2n+1}+d_{2n}), d_{2n}.d_{2n}/(d_{2n+1}+d_{2n})\}$,

We have the following cases to consider:

- Case-1:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = d_{2n}$, then from (3.1), we have $d_{2n+1} \ll q.d_{2n}$ implies $|d_{2n+1}| \leq q|d_{2n}|$.
- Case-2:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = d_{2n+1}d_{2n}/(d_{2n}+d_{2n+1})$, then from (3.1), $d_{2n+1} \ll q.d_{2n+1}d_{2n}/(d_{2n}+d_{2n+1})$, i.e., $d_{2n+1} \ll -(1-q).d_{2n}$, whence $|d_{2n+1}| \leq (1-q)|d_{2n}|$.
- Case-3:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = 0$, then from (3.1), $d_{2n+1} \ll q.0$ implies $|d_{2n+1}| \leq 0$.
- Case-4:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = 0$, then from (3.1), $d_{2n+1} \ll q.0$ implies $|d_{2n+1}| \leq 0$.
- Case-5:** If $\ddot{U}x_{2n}, x_{2n+1}(A, B, S, T) = (d_{2n+1}+d_{2n}).d_{2n}/(1+d_{2n})$, then from (3.1), $d_{2n+1} \ll q.d_{2n}^2$.

Case-2: If $\check{U}_{w, x_{2n+1}}(A, B, S, T) = d(Au, Su)d(y_{2n+1}, y_{2n})/[d(z, y_{2n})+d(Au, y_{2n+1})]$, then we have
 $d(Au, y_{2n+1}) \ll q \cdot d(Au, Su)d(y_{2n+1}, y_{2n})/[d(z, y_{2n})+d(Au, y_{2n+1})]$,
i.e., $|d(Au, y_{2n+1})| \leq q|d(Au, Su)d(y_{2n+1}, y_{2n})|/[d(z, y_{2n})+d(Au, y_{2n+1})]$
 Taking $n \rightarrow \infty$, it yields $|d(Au, z)| \leq q \cdot |d(Au, z)| \cdot 0/[d(Au, z)] = 0$. Thus $Au=z$.

Case-3: If $\check{U}_{w, x_{2n+1}}(A, B, S, T) = d(Bx_{2n+1}, z)d(Au, Tx_{2n+1})/[d(z, Tx_{2n+1})+d(Au, Bx_{2n+1})]$, then we have
 $d(Au, y_{2n+1}) \ll q \cdot d(y_{2n+1}, z)d(Au, y_{2n})/[d(z, y_{2n})+d(Au, y_{2n+1})]$,
i.e., $|d(Au, y_{2n+1})| \leq q|d(y_{2n+1}, z)d(Au, y_{2n})|/[d(z, y_{2n})+d(Au, y_{2n+1})]$.
 Taking $n \rightarrow \infty$, it yields $|d(Au, z)| \leq q \cdot 0 \cdot |d(Au, z)|/[d(Au, z)] = 0$, or, $|d(Au, z)| \leq 0$. Thus $Au=z$.

Case-4: If $\check{U}_{w, x_{2n+1}}(A, B, S, T) = d(Au, z)d(y_{2n+1}, z)/[d(Au, y_{2n+1})+d(z, y_{2n})]$, then we have
 $d(Au, y_{2n+1}) \ll q \cdot d(Au, z)d(y_{2n+1}, z)/[d(Au, y_{2n+1})+d(z, y_{2n})]$,
i.e., $|d(Au, y_{2n+1})| \leq q|d(Au, z)d(y_{2n+1}, z)|/[d(Au, y_{2n+1})+d(z, y_{2n})]$.
 Taking $n \rightarrow \infty$, it yields $|d(Au, z)| \leq q \cdot |d(Au, z)| \cdot 0/[d(Au, z)] = 0$, *i.e.*, $|d(Au, z)| \leq 0$. Thus $Au=z$.

Case-5: If $\check{U}_{w, x_{2n+1}}(A, B, S, T) = d(Au, y_{2n})d(y_{2n+1}, y_{2n})/[1+d(z, y_{2n})]$, then we have
 $d(Au, y_{2n+1}) \ll q \cdot d(Au, y_{2n})d(y_{2n+1}, y_{2n})/[1+d(z, y_{2n})]$
i.e., $|d(Au, y_{2n+1})| \leq q \cdot |d(Au, y_{2n})d(y_{2n+1}, y_{2n})|/[1+d(z, y_{2n})]$.
 Taking $n \rightarrow \infty$, it yields $|d(Au, z)| \leq q \cdot |d(Au, z)| \cdot 0/[1+0] = 0$, *i.e.*, $|d(Au, z)| \leq 0$. Thus $Au=z$.

Case-6: If $\check{U}_{w, x_{2n+1}}(A, B, S, T) = d(Au, z)d(y_{2n+1}, y_{2n})/[d(z, y_{2n})+d(Au, y_{2n+1})]$, then we have
 $d(Au, y_{2n+1}) \ll q \cdot d(Au, z)d(y_{2n+1}, y_{2n})/[d(z, y_{2n})+d(Au, y_{2n+1})]$,
i.e., $|d(Au, y_{2n+1})| \leq q|d(Au, z)d(y_{2n+1}, y_{2n})|/[d(z, y_{2n})+d(Au, y_{2n+1})]$,
 Taking $n \rightarrow \infty$, it yields $|d(Au, z)| \leq q \cdot |d(Au, z)| \cdot 0/[d(Au, z)] = 0$. Thus $Au=z$.

Case-7: If $\check{U}_{w, x_{2n+1}}(A, B, S, T) = d(z, y_{2n})d(Au, y_{2n+1})/[d(z, y_{2n})+d(Au, y_{2n+1})]$, then we have
 $d(Au, y_{2n+1}) \ll q \cdot d(z, y_{2n})d(Au, y_{2n+1})/[d(z, y_{2n})+d(Au, y_{2n+1})]$,
i.e., $|d(Au, y_{2n+1})| \leq q|d(z, y_{2n})d(Au, y_{2n+1})|/[d(z, y_{2n})+d(Au, y_{2n+1})]$,
 Taking $n \rightarrow \infty$, it yields $|d(Au, z)| \leq q \cdot 0 \cdot |d(Au, z)|/[d(Au, z)] = 0$. Thus $Au=z$.

Case-8: If $\check{U}_{w, x_{2n+1}}(A, B, S, T) = d(y_{2n+1}, z)d(z, y_{2n})/[d(Au, y_{2n+1})+d(z, y_{2n})]$, then we have
 $d(Au, y_{2n+1}) \ll q \cdot d(y_{2n+1}, z)d(z, y_{2n})/[d(Au, y_{2n+1})+d(z, y_{2n})]$,
i.e., $|d(Au, y_{2n+1})| \leq q|d(y_{2n+1}, z)d(z, y_{2n})|/[d(Au, y_{2n+1})+d(z, y_{2n})]$,
 Taking $n \rightarrow \infty$, it yields $|d(Au, z)| \leq q \cdot 0 \cdot 0/[d(Au, z)] = 0$. Thus $Au=z$.

Case-9: If $\check{U}_{w, x_{2n+1}}(A, B, S, T) = d(Au, z)d(z, y_{2n})/[d(Au, z)+d(z, y_{2n})]$, then we have
 $d(Au, y_{2n+1}) \ll q \cdot d(Au, z)d(z, y_{2n})/[d(Au, z)+d(z, y_{2n})]$,
i.e., $|d(Au, y_{2n+1})| \leq q|d(Au, z)d(z, y_{2n})|/[d(Au, z)+d(z, y_{2n})]$,
 Taking $n \rightarrow \infty$, it yields $|d(Au, z)| \leq q \cdot |d(Au, z)| \cdot 0/[d(Au, z)] = 0$. Thus $Au=z$.

Case-10: If $\check{U}_{w, x_{2n+1}}(A, B, S, T) = \frac{1}{2}sd(z, y_{2n})d(y_{2n+1}, y_{2n})/[d(Au, y_{2n+1})+d(z, y_{2n})]$, then we have
 $d(Au, y_{2n+1}) \ll \frac{1}{2}qs \cdot d(z, y_{2n})d(y_{2n+1}, y_{2n})/[d(Au, y_{2n+1})+d(z, y_{2n})]$
i.e., $|d(Au, y_{2n+1})| \leq \frac{1}{2}qs|d(z, y_{2n})d(y_{2n+1}, y_{2n})|/[d(Au, y_{2n+1})+d(z, y_{2n})]$
 Taking $n \rightarrow \infty$, it yields $|d(Au, z)| \leq \frac{1}{2}qs \cdot 0 \cdot 0/[d(Au, z)] = 0$, or, $|d(Au, z)| \leq 0$. Thus $Au=z$.

Hence in all cases $Au=z=Su$, *i.e.*, $u \in X$ is a coincidence point of (A, S) . Further, since $z=Au$ and $AX \subseteq TX$, there exists $v \in X$ such that $Au=Tv$. We claim that $Bv=z$. If not, then putting $x=u, y=v$ in condition (ii), and using $Au=Su=Tv=z$, we have

$$d(z, Bv) = d(Au, Bv) \ll q \cdot \check{U}_{w, v}(A, B, S, T) \tag{3.10}$$

where $\check{U}_{w, v}(A, B, S, T) \in \{d(Su, Tv), d(Au, Su)d(Bv, Tv)/[d(Su, Tv)+d(Au, Bv)], d(Bv, Su)d(Au, Tv)/[d(Su, Tv)+d(Au, Bv)], d(Au, Su)d(Bv, Su)/[d(Au, Bv)+d(Su, Tv)], d(Au, Tv)d(Bv, Tv)/[1+d(Su, Tv)], d(Au, Su)d(Bv, Tv)/[d(Su, Tv)+d(Au, Bv)],$

$d(Su, Tv)d(Au, Bv)/[d(Su, Tv)+d(Au, Bv)], d(Bv, Su)d(Su, Tv)/[d(Au, Bv)+d(Su, Tv)], d(Au, Su)d(Su, Tv)/[d(Au, Su)+d(Su, Tv)], \frac{1}{2}sd(Su, Tv)d(Bv, Tv)/[d(Au, Bv)+d(Su, Tv)]\}$,

$$= \{0, 0, 0, 0, 0, 0, d(z, z)d(z, z)/2d(z, z), \frac{1}{2}s \cdot d(z, z)d(Bv, z)/[d(z, Bv)+d(z, z)]\} = \{0, \frac{1}{2}d(z, z)\} = 0.$$

Thus $Au = Su = z = Bv = Tv$, *i.e.*, u and v are coincidence points of (A, S) and (B, T) respectively. These two pairs are weakly compatible therefore, from condition (iii), we have $ASu=SAu=Az=Sz$ and $BTv=TBv=Bz=Tz$. We now claim that $Az=Bz$. If not, then putting $x=z, y=z$ in condition (ii), we have

$$d(Az, Bz) \ll q \check{U}_{z, z}(A, B, S, T) \tag{3.11}$$

where $\check{U}_{z,z}(A,B, S, T) \in \{d(Sz,Tz), d(Az, Sz)d(Bz,Tz)/[d(Sz,Tz)+d(Az,Bz)], d(Bz,Sz)d(Az,Tz)/[d(Sz,Tz)+d(Az,Bz)], d(Az,Sz)d(Bz,Sz)/[d(Az, Bz)+d(Sz,Tz)], d(Az,Tz)d(Bz,Tz)/[1+d(Sz,Tz)], d(Az,Sz)d(Bz,Tz)/[d(Sz,Tz)+d(Az,Bz)], d(Sz,Tz)d(Az,Bz)/[d(Sz,Tz)+d(Az, Bz)], d(Bz,Sz)d(Sz,Tz)/[d(Az, Bz)+d(Sz,Tz)], d(Az,Sz)d(Sz,Tz)/[d(Az,Sz)+d(Sz,Tz)], \frac{1}{2}s.d(Sz,Tz)d(Bz,Tz)/[d(Az,Bz)+d(Sz,Tz)]\}$,

$$= \{0, \frac{1}{2}d(Az,Bz), 0, 0,0, \frac{1}{2}d(Az,Bz), \frac{1}{2} d(Az, Bz), 0, 0\}.$$

If 0 is chosen then $d(Az, Bz) \ll q.0$ yields $Az=Bz$. Also, if $\frac{1}{2}d(Az, Bz)$ is chosen then (3.11) implies $d(Az,Bz) \ll \frac{1}{2}q d(Az, Bz)$, i.e., $|d(Az,Bz)| \leq \frac{1}{2}q \cdot |d(Az, Bz)|$, a contradiction. Thus $Az=Bz$. Therefore z is a coincidence point of A, B, S and T .

We now claim that z is a common fixed point of these four mappings. For, putting $x=u, y=z$ in condition (ii), and using $Au=Su=z, Bz=Tz$, we have

$$d(z, Bz)=d(Au,Bz) \ll q\check{U}_z, z (A, B, S, T), \tag{3.12}$$

where $\check{U}_{u,z}(A,B, S, T) \in \{d(Su,Tz), d(Au, Su)d(Bz,Tz)/[d(Su,Tz)+d(Au,Bz)], d(Bz,Su)d(Au,Tz)/[d(Su,Tz)+d(Au,Bz)], d(Au,Su)d(Bz,Su)/[d(Au, Bz)+d(Su,Tz)], d(Au,Tz)d(Bz,Tz)/[1+d(Su,Tz)], d(Au,Su)d(Bz,Tz)/[d(Su,Tz)+d(Au,Bz)], d(Su,Tz)d(Au,Bz)/[d(Su,Tz)+d(Au, Bz)], d(Bz,Su)d(Su,Tz)/[d(Au, Bz)+d(Su,Tz)], d(Au,Su)d(Su,Tz)/[d(Au,Su)+d(Su,Tz)], \frac{1}{2}s.d(Su,Tz)d(Bz,Tz)/[d(Au,Bz)+d(Su,Tz)]\}$,

$$= \{d(z,Bz), 0, \frac{1}{2}d(Bz,z), 0, 0,0, \frac{1}{2}d(z,Bz), \frac{1}{2}.d(Bz,z), 0, \frac{1}{2}s.d(z,Bz)\}=\{d(z,Bz), 0, \frac{1}{2}s.d(z,Bz)\}.$$

If $d(z, Bz)$ is chosen, then $d(z, Bz) \ll q. d(z, Bz)$ implies $|d(z, Bz)| \leq q \cdot |d(z, Bz)|$, a contradiction, thus $Bz=z$. If $\frac{1}{2}d(z, Bz)$ is chosen, then $d(z, Bz) \ll \frac{1}{2}qd(z, Bz)$ implies $|d(z, Bz)| \leq \frac{1}{2}q|d(z, Bz)|$, a contradiction, thus $Bz=z$. If 0 is chosen then also, $Bz=z$. Therefore z is a common fixed point of A, B, S and T . This common fixed point is unique. If not, then assume $w \in X$ be another common fixed point such that $z \neq w$. Now, by putting $x=z$ and $y=w$ in condition (ii) and using $Az=Sz=z, Bw=Tw=w$, we get a contradiction. Thus z is the unique common fixed point of A, B, S and T . This completes the proof.

If $A=B=f$ and $S=T=g$ then Theorem 3.1 reduces to following corollary:

Corollary 3.2: Let (X, d) be a complete complex valued b-metric space with metric parameter $s \geq 1$ and $f, g: X \rightarrow X$ be self-mappings on X . Suppose that the set $f(X)$ is closed, and $f(X)$ subseteq $g(X)$. If the following conditions satisfy:

$$\begin{aligned} d(fx, fy) \ll q \cdot \check{U}_x, y (f, g), \text{ for all } x, y \in X, \text{ where } q \text{ is a non-negative real number such that } 0 \leq q < 1/(s^2+s) \tag{3.12} \\ \text{where } \check{U}_{x,y}(f,g) \in \{d(gx,gy), d(fx, gx)d(fy,gy)/[d(gx,gy)+d(fx,fy)], d(fy,gx)d(fx,gy)/[d(gx,gy)+d(fx,fy)], \\ d(fx,gx)d(fy,gx)/[d(fx, fy)+d(gx,gy)], d(fx,gy)d(fy,gy)/[1+d(gx,gy)], d(fx,gx)d(fy,gy)/[d(gx,gy)+d(fx,fy)], \\ d(gx,gy)d(fx,fy)/[d(gx,gy)+d(fx, fy)], d(fy,gx)d(gx,gy)/[d(fx, fy)+d(gx,gy)], d(fx,gx)d(gx,gy)/[d(fx,gx)+d(gx,gy)], \\ \frac{1}{2}s.d(gx,gy)d(fy,gy)/[d(fx,fy)+d(gx,gy)]\}. \end{aligned}$$

If the pair (f, g) is weakly compatible, then A, B, S and T have unique common fixed point in X .

If we put $S=T=I$, the identity mapping, then condition (i) of Theorem 3.1 reduces to $A(X)$ subseteq $X, B(X)$ subseteq X , which is always true. Further, the weakly compatibility reduces to $AIx=IAx$, and $BIx=IBx$, where $Ix=x$; thus weakly compatibility condition always hold. Therefore, Theorem 3.1 reduces to following corollary:

Corollary 3.2: Let (X, d) be a complete complex valued b-metric space and $A, B: X \rightarrow X$ be self-mappings of (X, d) with metric parameter $s \geq 1$. Suppose that the set $B(X)$ is closed. If the following inequality:

$$\begin{aligned} d(Ax, By) \ll q\check{U}_{x,y}(A, B), \text{ for all } x, y \in X, \text{ where } q \text{ is a non-negative real number such that } 0 \leq q < 1/(s^2+s) \tag{3.13} \\ \text{where } \check{U}_{x,y}(A,B) \in \{d(x,y), d(Ax, x)d(By,y)/[d(x,y)+d(Ax,By)], d(By,x)d(Ax,y)/[d(x,y)+d(Ax,By)], d(Ax,x)d(By,x)/[d(Ax, \\ By)+d(x,y)], d(Ax,y)d(By,y)/[1+d(x,y)], d(Ax,x)d(By,y)/[d(x,y)+d(Ax,By)], d(x,y)d(Ax,By)/[d(x,y)+d(Ax, By)], \\ d(By,x)d(x,y)/[d(Ax, By)+d(x,y)], d(Ax,x)d(x,y)/[d(Ax,x)+d(x,y)], \frac{1}{2}s.d(x,y)d(By,y)/[d(Ax,By)+d(x,y)]\}, \\ \text{then } A,B, S \text{ and } T \text{ have unique common fixed point in } X. \end{aligned}$$

Remark 3.3: Corollary 3.2 generalizes Banach contraction mapping principal [5] in complex-valued b-metric space, if only first co-ordinate is chosen. Also, since $q < 1/2$, the sum of product of first co-ordinate with λ and product of fifth co-ordinate with μ , will produce the inequality of Azam *et al.* [2] with $\lambda + \mu < 1$. Hence we have generalized [2] in complex-valued b-metric spaces.

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