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ALTERING DISTANCE OF TWO VARIABLES AND ITS APPLICATION

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ABSTRACT

In this paper, we have obtained a number of fixed point theorems for self mappings defined on a complete non-Archimedean Menger probabilistic metric space satisfying a contractive inequality by using the concept of altering distance of two variables and non expansive mappings.

INTRODUCTION:

Result on the existence of fixed points for self mappings on a non- Archimedean probabilistic metric space have been obtained by Hadžić [74], Istrătescu [92], Cho, Park and Chang [32] and Cho, Ha and Chang [35] others. In 1994 Chang, Cho, and Kang [25] summarized almost all the results in "*Fixed point theorem for single valued mappings in some special probabilistic metric spaces*" and defined non- Archimedean Menger space of type C_g .

Khan Swaleh and Sessa introduced the notion of altering distance [7] and proved fixed point theorems involving altering distance. Using the idea of altering distance, Binayak S. Choudhury and P. N. Dutta [2] introduced the notion of function of two variables of type *A*. The concept of function of type *A* and the results of the above authors prompted us to go for further generalization of the above results in non – Archimedean Menger probabilistic metric space of type C_g involving a two variable function of type *A*. We have also defined non-expansive self-mapping with respect to φ, where $\varphi:[0,\infty)\to[0,\infty)$ is a mapping is satisfying certain condition given in section 3 and using this concept we have also obtained some results on fixed point.

PRELIMINARIES:

In this section we recall some basic definitions and results of non- Archimedean probabilistic metric space. For more details we refer the reader to [1], [4], [5] and [6].

Definition: 1.1 [1] A non- Archimedean probabilistic metric space is an ordered pair (*X*, *F*), X a non empty set and $F: X\times X \to L$ a map such that,

$$(i)F_{p,q}(x)=1\ ,\forall\ x>0 \text{ iff } p=q,$$

$$(ii)F_{p,q}(0)=0,$$

$$(iii)F_{p,q}(x)=F_{q,p}(x),$$

$$\text{and}\quad (iv)F_{p,q}(x)=1\ ,\ F_{q,r}(y)=1 \Rightarrow F_{p,r}(\max\{x,y\})=1,$$

where *L* is the set of all distribution functions and $F_{p,q}(x)$ is the value of the function $F_{p,q}=F(p,q)\in L$ at $x\in R$.

Definition: 1.2 [1], [4] A non- Archimedean Menger probabilistic metric space is a triple (*X*, *F*, *t*), where (*X*, *F*) is a non- Archimedean probabilistic metric space and *t* is a t- norm such that,

$F_{p,r}(\max\{x,y\}\geq t(F_{p,q}(x),F_{q,r}(y))\ \forall\ p,q,r\in X$ and $x,y\geq 0$.

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Definition: 1.3 Let $\Omega = \{g : g : [0,1] \to [0,\infty]$ is continuous, strictly decreasing s.t $g(1) = 0, g(0) < \infty\}$ is a set of functions. A probabilistic metric space is said to be of type C_g if $g \in \Omega$ such that,

$$g\left(F_{p,q}(x)\right) \leq g\left(F_{p,r}(x)\right) + g\left(F_{r,q}(x)\right) \ \forall\ p,q,r \in X \text{ and } x \geq 0.$$

NOTE: Throughout this paper we consider (*X, F, t*) a complete non- Archimedean Menger probabilistic metric space of type C_g .

In 1984 M.S. Khan, M. Swaleh and S. Sessa [7] gave the notion of altering distance between two points and proved fixed point theorems using this altering distance.

Definition: 1.4 [7] An altering distance is a mapping $\phi : [0,\infty) \to [0,\infty)$ Such that,

(a) ϕ is increasing and continuous, and
(b) $\phi(x) = 0$ iff $x = 0$.

Choudhury and Dutta [2] extended this idea to the functions of two variables which satisfy the condition of the type *A*. Following is the definition and example given in [2] which may be termed as altering distance of two variables.

Definition: 1.5 [2] A function $\phi : R^+ \times R^+ \to R^+$ is said to be of the type *A* if

(a) ϕ is continuous and monotonic increasing in both the arguments, and
(b) $\phi(0,0) = 0$ and $\phi(\varepsilon,0) = 0 \Rightarrow \varepsilon = 0$.

In the above definition the term used "monotonic increasing" in both the arguments Means

$a \leq b \Rightarrow \phi(a,c) \leq \phi(b,c)$ and $a \leq b \Rightarrow \phi(c,a) \leq \phi(c,b)$, for all *c*.

Following Remark and examples are given in [2].

Remark: 1.1 $\phi(\varepsilon,\varepsilon) = 0 \ \Rightarrow \varepsilon = 0$ because, $\phi(\varepsilon,0) \leq \phi(\varepsilon,\varepsilon) = 0 \ \Rightarrow \phi(\varepsilon,0) = 0 \Rightarrow \varepsilon = 0.$

Example: 1.1 $\phi : R^+ \times R^+ \to R^+$ is defined as,

$$(i)\ \ \phi(a,b) = (a^p + b^q)^k,$$

$$(ii)\ \ \phi(a,b) = a^p.b^q + a^k,$$

where *p, q* and *k* are positive integers. Then ϕ is altering distance of two variables. Following lemma proved by Chang, [1] has been used by us in proving our results.

Lemma: 1.1 [4] Let $\{p_n\}$ be a sequence in *X* such that $\lim_{n\to\infty} F_{p_n,p_{n+1}}(x) = 1$ for all $x > 0$. If the sequence $\{p_n\}$ is not a Cauchy sequence in *X*, then there exists $\varepsilon_0 > 0,\ t_0 > 0$ and two sequence $\{m_i\}$ and $\{n_i\}$ of positive integers such that,

(i) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$,

(ii) $g(F_{p_{m_i},p_{n_i}}(t_0)) > g(1-\varepsilon_0)$ and $g(\ F_{p_{m_i-1},p_{n_i}}(t_0)) \leq g(1-\varepsilon_0)$.

Remark: 1.2 If sequence $\{p_n\}$ is not a Cauchy sequence and $\lim_{n\to\infty} g(F_{p_n,p_{n+1}}(x)) = 0.$

Then,

$$g(1-\varepsilon_0) < g(F_{p_{m_i},p_{n_i}}(t_0)) \le g(F_{p_{m_i},p_{m_{i-1}}}(t_0)) + g(F_{p_{m_{i-1}},p_{n_i}}(t_0)) . \text{ and } \lim_{i\to\infty} g(F_{p_{m_i},p_{n_i}}(t_0)) = g(1-\varepsilon_0). \tag{1.1}$$

Again,

$$g(F_{p_{m_i},p_{n_i}}(t_0)) \le g(F_{p_{m_i},p_{m_{i-1}}}(t_0)) + g(F_{p_{m_{i-1}},p_{n_{i-1}}}(t_0)) + g(F_{p_{n_{i-1}},p_{n_i}}(t_0))$$

and

$$g(F_{p_{m_{i-1}},p_{n_{i-1}}}(t_0)) \le g(F_{p_{m_{i-1}},p_{m_i}}(t_0)) + g(F_{p_{m_i},p_{n_i}}(t_0)) + g(F_{p_{n_i},p_{n_{i-1}}}(t_0)) .$$

Hence,

$$\lim_{i\to\infty} g(F_{p_{m_{i-1}},p_{n_{i-1}}}(t_0)) = g(1-\varepsilon_0) \tag{1.2}$$

Also,

$$g(F_{p_{n_{i-1}},p_{m_{i+1}}}(t_0)) \le g(F_{p_{n_{i-1}},p_{n_i}}(t_0)) + g(F_{p_{n_i},p_{m_i}}(t_0)) + g(F_{p_{m_i},p_{m_{i+1}}}(t_0))$$

and

$$g(F_{p_{m_i},p_{n_i}}(t_0)) \le g(F_{p_{n_i},p_{n_{i-1}}}(t_0)) + g(F_{p_{n_{i-1}},p_{m_{i+1}}}(t_0)) + g(F_{p_{m_{i+1}},p_{m_i}}(t_0)) .$$

Therefore, from (1), (2) we have,

$$\lim_{i\to\infty} g(F_{p_{n_{i-1}},p_{m_{i+1}}}(t_0)) = g(1-\varepsilon_0) \ldots \tag{1.3}$$

Lastly,

$$g(F_{p_{m_i},p_{n_i}}(t_0)) \le g(F_{p_{m_i},p_{n_{i-1}}}(t_0)) + g(F_{p_{n_{i-1}},p_{n_i}}(t_0))$$

and

$$g(F_{p_{m_i},p_{n_{i-1}}}(t_0)) \le g(F_{p_{m_i},p_{n_i}}(t_0)) + g(F_{p_{n_i},p_{n_{i-1}}}(t_0)) .$$

Hence from (1), (2) we have

$$\lim_{i\to\infty} g(F_{p_{m_i},p_{n_{i-1}}}(t_0)) = g(1-\varepsilon_0) \tag{1.5}$$

2. MAIN RESULTS

In this section a unique fixed point theorem in complete non–Archimedean Menger space for a class of self mappings satisfying certain inequality involving function of two variables has been derived.

Theorem: 2.1 Suppose (X, F; t) is a complete non- Archimedean Menger probabilistic metric space. Let $T : X \to X$ be a self mapping on X satisfying,

$$\phi\Big(g(F_{Tp,Tq}(x), g(F_{p,Tp}(x)\Big) + \phi\Big(g(F_{q,Tq}(x), g(F_{q,T^2p}(x)\Big)$$
$$\le \phi\Big(g(F_{p,q}(x), g(F_{p,Tp}(x)\Big) + \beta\phi\Big(g(F_{q,Tq}(x), g(F_{q,Tp}(x)\Big) \tag{2.1}$$

where $0 < \alpha < 1$, $0 < \beta \le 1$ and ϕ is a function of the type A. Then there exists unique fixed point of T in X.

Proof: In fact using the condition (1) we shall get a Cauchy sequence in X whose limit point will turn out to be unique fixed point of T. Following is the procedure for finding the Cauchy sequence in X.

Picking $p_0 \in X$, we construct a sequence $\{p_n\}$ inductively as

$p_n = Tp_{n-1}$, (n = 1, 2, 3 . . .).

Putting $q = Tp$ in (2.1) we get,

$$\phi\Big(g(F_{Tp,T^2p}(x), g(F_{p,Tp}(x)\Big)+\phi\Big(g(F_{Tp,T^2p}(x), g(F_{Tp,T^2p}(x)\Big)$$
$$\le \alpha\phi\Big(g(F_{p,Tp}(x), g(F_{p,Tp}(x)\Big)+\beta\Big(g(F_{Tp,T^2p}(x), g(F_{Tp,Tp}(x)\Big) \tag{2.2}$$

Since, $0<\beta\le 1$ hence from (3) we get,

$$\beta\phi\Big(g(F_{Tp,T^2p}(x),0\Big)\le \beta\phi\Big(g(F_{Tp,T^2p}(x), g(F_{Tp,T^2p}(x)\Big)\le \phi\Big(g(F_{Tp,T^2p}(x), g(F_{Tp,T^2p}(x)\Big),$$

hence,

$$\phi\Big(g(F_{Tp,T^2p}(x), g(F_{p,Tp}(x)\Big)\le \alpha\phi\Big(g(F_{p,Tp}(x), g(F_{p,Tp}(x)\Big)$$
$$\le \phi\Big(g(F_{p,Tp}(x), g(F_{p,Tp}(x)\Big) \tag{2.3}$$

i.e. $\phi\Big(g(F_{Tp,T^2p}(x), g(F_{p,Tp}(x)\Big)\le \phi\Big(g(F_{p,Tp}(x), g(F_{p,Tp}(x)\Big)$ (2.4)

or $g(F_{Tp,T^2p}(x))\le g(F_{p,Tp}(x))$ (2.5)

Putting $p=p_{n-1}$ in (2.5) we get,

$$g(F_{p_n,p_{n+1}}(x))\le g(F_{p_{n-1},p_n}(x)) \tag{2.6}$$

This shows that the sequence $\{g(F_{p_n,p_{n+1}}(x))\}$ is monotonic decreasing and bounded below. So, $g(F_{p_n,p_{n+1}}(x))\to a\in X$ as $n\to\infty$ (2.7)

Again putting $p=p_{n-1}$ in (2.4) we get,

$$\phi(g(F_{p_n,p_{n+1}}(x), g(F_{p_{n-1},p_n}(x))\le \alpha\phi(g(F_{p_{n-1},p_n}(x), g(F_{p_{n-1},p_n}(x)).$$

Making $n\to\infty$ and using continuity of ϕ we get,

$\phi(a,a)\le \alpha\phi(a,a).$

Since $0<\alpha<1,$ therefore $\phi(a,a)=0$, otherwise, $\phi(a,a)<\phi(a,a)$, so from the remark 1.1, $a=0$.Therefore,

$$\lim_{n\to\infty} g(F_{p_n,p_{n+1}}(x))=0. \tag{2.8}$$

Next we show that $\{p_n\}$ is a Cauchy sequence. Suppose, on the contrary, $\{p_n\}$ is not a Cauchy sequence. Then by Lemma 1.1 there exists $\varepsilon_0>0$, $t_0>0$ and sets of positive integer $\{m_i\},\{n_i\}$ such that,

$$(i)\ m_i > n_i+1 \text{ and } n_i\to\infty \text{ as } i\to\infty,$$
$$(ii)\ g(F_{p_{m_i},p_{n_i}}(t_0))>g(1-\varepsilon_0) \text{ and } g(F_{p_{m_{i-1}},p_{n_i}}(t_0)\le g(1-\varepsilon_0). \tag{2.9}$$

Now,

$$g(1-\varepsilon_0)<g(F_{p_{m_i},p_{n_i}}(t_0))\le g(F_{p_{m_i},p_{m_{i-1}}}(t_0))+g(F_{p_{m_{i-1}},p_{n_i}}(t_0)).$$

Making $i\to\infty$,

$$\lim_{i\to\infty} g(F_{p_{m_i},p_{n_i}}(t_0))=g(1-\varepsilon_0) \tag{2.10}$$

Again,

$g(F_{p_{m_i},p_{n_i}}(t_0)) \le g(F_{p_{m_i},p_{m_{i-1}}}(t_0)) + g(F_{p_{m_{i-1}},p_{n_{i-1}}}(t_0)) + g(F_{p_{n_{i-1}},p_{n_i}}(t_0))$

and

$g(F_{p_{m_{i-1}},p_{n_{i-1}}}(t_0)) \le g(F_{p_{m_{i-1}},p_{m_i}}(t_0)) + g(F_{p_{m_i},p_{n_i}}(t_0)) + g(F_{p_{n_i},p_{n_{i-1}}}(t_0))$.

Hence

$$\lim_{i\to\infty} g(F_{p_{m_{i-1}},p_{n_{i-1}}}(t_0)) = g(1-\varepsilon_0) \tag{2.11}$$

Also,

$g(F_{p_{n_{i-1}},p_{m_{i+1}}}(t_0)) \le g(F_{p_{n_{i-1}},p_{n_i}}(t_0)) + g(F_{p_{n_i},p_{m_i}}(t_0)) + g(F_{p_{m_i},p_{m_i+1}}(t_0))$

and

$g(F_{p_{m_i},p_{n_i}}(t_0)) \le g(F_{p_{n_i},p_{n_{i-1}}}(t_0)) + g(F_{p_{n_{i-1}},p_{m_{i+1}}}(t_0)) + g(F_{p_{m_{i+1}},p_{m_i}}(t_0))$.

Therefore from (2.8), (2.10) we have,

$$\lim_{i\to\infty} g(F_{p_{n_{i-1}},p_{m_{i+1}}}(t_0)) = g(1-\varepsilon_0) \tag{2.12}$$

Thus,

$g(F_{p_{m_i},p_{n_i}}(t_0)) \le g(F_{p_{m_i},p_{n_{i-1}}}(t_0)) + g(F_{p_{n_{i-1}},p_{n_i}}(t_0))$

and

$g(F_{p_{m_i},p_{n_{i-1}}}(t_0)) \le g(F_{p_{m_i},p_{n_i}}(t_0)) + g(F_{p_{n_i},p_{n_{i-1}}}(t_0))$.

Making $i \to \infty$ and noting (2.8), (2.10), we have,

$$\lim_{i\to\infty} g(F_{p_{m_i},p_{n_{i-1}}}(t_0)) = g(1-\varepsilon_0) \tag{2.13}$$

Now putting $p = p_{m_{i-1}}$, $q = p_{n_{i-1}}$ and $x = t_0 > 0$ in (2.1) we get,

$$\phi(g(F_{p_{m_i},p_{n_i}}(t_0)), g(F_{p_{m_{i-1}},p_{m_i}}(t_0))) + \phi(g(F_{p_{n_{i-1}},p_{n_i}}(t_0)), g(F_{p_{n_{i-1}},p_{m_{i+1}}}(t_0)))$$

$$\le \alpha\phi(g(F_{p_{m_{i-1}},p_{n_{i-1}}}(t_0)), g(F_{p_{m_{i-1}},p_{m_i}}(t_0))) + \beta\phi(g(F_{p_{n_{i-1}},p_{n_i}}(t_0)), g(F_{p_{n_{i-1}},p_{m_i}}(t_0))).$$

Making $i \to \infty$ using (8), (10) – (13) and the continuity of ϕ, we get,

$$\phi(g(1-\varepsilon_0),0)) + \phi(0, g(1-\varepsilon_0))) \le \alpha\phi(g(1-\varepsilon_0),0)) + \beta\phi(0, g(1-\varepsilon_0))).$$

Since $0 < \beta \le 1$, $\beta\phi(0, g(1-\varepsilon_0))) \le \phi(0, g(1-\varepsilon_0))$, it follows that,

$$\phi(g(1-\varepsilon_0),0)) \le \alpha\phi(g(1-\varepsilon_0),0)).$$

This is possible only when $\phi(g(1-\varepsilon_0),0)) = 0$ (follows from the condition on α, ε and g). Therefore $\{p_n\}$ is a Cauchy sequence.

Since $(X, F; t)$ is complete so $\{p_n\} \to z \in X$. Putting $q = z$, $p = p_n$ in (1), we get,

$$\phi\Big(g(F_{p_{n+1},Tz}(x)), g(F_{p_n,p_{n+1}}(x))\Big) + \phi\Big(g(F_{z,Tz}(x)), g(F_{z,p_{n+2}}(x))\Big)$$

$$\le \alpha\phi\Big(g(F_{p_n,z}(x)), g(F_{p_n,p_{n+1}}(x))\Big) + \beta\phi\Big(g(F_{z,Tz}(x)), g(F_{z,p_{n+1}}(x))\Big).$$

Making $i \to \infty$,

$$\phi\big(g(F_{z,Tz}(x)),0\big)+\phi\big(g(F_{z,Tz}(x)),0\big)$$

$$\leq \alpha\phi(0,0)+\beta\phi\big(g(F_{z,Tz}(x)),0\big).$$

Since $0<\beta\leq 1$, so, $\phi\big(g(F_{z,Tz}(x)),0\big)\leq \alpha\phi(0,0)=0$

i. e. $\phi\big(g(F_{z,Tz}(x)),0\big)=0$ which implies that $g(F_{z,Tz}(x))=0$.

Therefore, $z=Tz$. Hence *z* is a fixed point of *T*.

For uniqueness, suppose z_1, z_2 are fixed points of T i.e. $Tz_1=z_1$ and $Tz_2=z_2$.

Putting $p=z_1$ and $q=z_2$ in (1) we get,

$$\phi\big(g(F_{z_1,z_2}(x)), g(F_{z_1,z_1}(x))\big)+\phi\big(g(F_{z_2,z_2}(x)), g(F_{z_2,z_1}(x))\big)$$

$$\leq \alpha\phi\big(g(F_{z_1,z_2}(x)), g(F_{z_1,z_2}(x))\big)+\beta\phi\big(g(F_{z_2,z_2}(x)), g(F_{z_2,z_1}(x))\big),$$

i.e.

$$\phi\big(g(F_{z_1,z_2}(x)),0\big)+\phi\big(0, g(F_{z_2,z_1}(x))\big)$$

$$\leq \alpha\phi\big(g(F_{z_1,z_2}(x)),0\big)+\beta\phi\big(0, g(F_{z_2,z_1}(x))\big).$$

Since $0<\beta\leq 1$ so we get $\phi\big(g(F_{z_1,z_2}(x)),0\big)\leq \alpha\phi\big(g(F_{z_1,z_2}(x)),0\big)$. Since $0<\alpha<1$,

hence $\phi\big(g(F_{z_1,z_2}(x)),0\big)<\phi\big(g(F_{z_1,z_2}(x)),0\big)$. This is possible only when $z_1=z_2$.

Therefore *T* has unique fixed point.

3. MAIN RESULTS

In this section we have introduced the concept of non- expansiveness of a self mapping with respect to φ, using which we have proved two theorems in non- Archimedean Menger probabilistic metric space of type C_g.

Definition: 3.1 Suppose (*X*, *F*; *t*) is a non- Archimedean probabilistic space and *f* is a self mapping defined on *X*. Then *f* is said to be non expansive with respect to φ if $g(F_{fp,fq}(x))\leq \varphi(g(F_{p,q}(x)))$, $\forall\ x>0$, where φ is a function on $[0,\infty)$ such that it is upper semi continuous from the right and $\varphi(t)<t$ for all $t>0$ [4].

We need the following Lemma proved by Cho, Ha and Chang [2].

Lemma: 3.1 [1], [4]. If $\varphi:[0,\infty)\to[0,\infty)$ is a function such that φ is upper semi continuous from the right and $\varphi(t)<t$ for all $t>0$, then

(a) For all $t\geq 0$, $\lim_{n\to\infty}\varphi^n(t)=0$, where $\varphi^n(t)$ is the n-th iteration of $\varphi(t)$.

(b) If $\{t_n\}$ is a non-decreasing sequence of real numbers and

$$t_{n+1}\leq \varphi(t_n),\ n=1,2,\ \ldots,$$

then $\lim_{n\to\infty} t_n = 0$. In particular, if $t \le \varphi(t)$ for all $t \ge 0$, then $t = 0$.

Theorem: 3.1 Suppose $(X, F; t)$ is a non- Archimedean Menger probabilistic metric space of type C_g and *f* a self mapping on *X* such that *f* is non expansive with respect to φ. Then there exists a unique fixed point of *f*.

Proof: Picking $P_0 \in X$, we construct a sequence $\{P_n\}$ inductively as $P_n = fp_{n-1}$.

Now,

$$g(F_{p_n, p_{n+1}}(x)) = g(F_{fp_{n-1}, fp_n}(x)) \le \varphi\big(g(F_{p_{n-1}, p_n}(x)\big).$$

Hence by lemma 3.1,

$$\lim_{n\to\infty} g(F_{p_n, p_{n+1}}(x)) = 0.$$

Next we prove that $\{P_n\}$ is a Cauchy sequence.

If, on the contrary, $\{P_n\}$ is not a Cauchy sequence, then by Lemma 1.1 there exists $\varepsilon_0 > 0$, $t_0 > 0$ and sets of positive integer $\{m_i\}, \{n_i\}$ such that,

(i) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$

(ii) $g(F_{p_{m_i}, p_{n_i}}(t_0)) > g(1-\varepsilon_0)$ and $g(\, F_{p_{m_i-1}, p_{n_i}}(t_0) \le g(1-\varepsilon_0)$

Now since $\quad g(F_{p_{m_i}, p_{n_i}}(t_0)) \le g(F_{p_{m_i}, p_{m_i-1}}(t_0)) + g(F_{p_{m_i-1}, p_{n_i}}(t_0)),$

hence,

$$\lim_{n\to\infty} g(F_{p_{m_i}, p_{n_i}}(t_0)) = g(1-\varepsilon_0).$$

Again,

$$g(1-\varepsilon_0) < g(F_{p_{m_i} p_{n_i}}(t_0)) \le g(F_{p_{n_i} p_{n_i+1}}(t_0)) + g(F_{p_{n_i+1} p_{m_i}}(t_0)),$$

making $i \to \infty$,

$$\lim_{n\to\infty} g(F_{p_{m_i} p_{n_i+1}}(t_0)) = g(1-\varepsilon_0)$$

i.e. $g(F_{p_{m_i} p_{n_i+1}}(t_0)) = g(F_{p_{n_i} p_{m_i-1}}(t_0)) \le \varphi(g(F_{p_{n_i} p_{m_i-1}}(t_0))$

again making $i \to \infty$,

$$g(1-\varepsilon_0) \le \varphi(g(1-\varepsilon_0)) < g(1-\varepsilon_0)$$

i.e. $g(1-\varepsilon_0) < g(1-\varepsilon_0)$, a contradiction.

Hence, $\{P_n\}$ is a Cauchy sequence. Since $(X, F; t)$ is complete so $\{p_n\} \to z \in X$.

Now,

$$g(F_{fz, p_{n+1}}(x)) = g(F_{fz, fp_n}(x)) \le \varphi(g(F_{z, p_n}(x))) \text{ as } n \to \infty$$

$$g(F_{fz,z}(x)) \le \varphi(g(F_{z,z}(x))),$$

i.e. $g(F_{fz,z}(x)) = 0$, so $z = fz$. Thus z is a fixed point of f. For uniqueness, suppose z, z' are fixed points of f. Then

$$g(F_{z,z'}(x)) = g(F_{fz,fz'}(x)) \le \varphi(g(F_{z,z'}(x))),$$

i.e. $g(F_{z,z'}(x)) < g(F_{z,z'}(x))$, which is possible only if $z = z'$. This proves the uniqueness.

Theorem: 3.2 Let $(X, F; t)$ be a complete non- Archimedean Menger probabilistic metric space and $f : X \to X$ a self mapping satisfying for all $p, q \in X$,

$$g(F_{fp,fq}(x)) \le \varphi(\max\{g(F_{p,q}(x)), g(F_{p,fp}(x)), g(F_{q,fq}(x)), g(F_{q,fp}(x))\}), \forall x > 0.$$

Then f has a unique fixed point.

Proof: Picking $p_0 \in X$, we construct a sequence $\{P_n\}$ inductively as $P_n = fp_{n-1}$,

n = 1, 2… Then,

$$g(F_{p_n,p_{n+1}}(x)) \le \varphi(\max\{g(F_{p_{n-1},p_n}(x)), g(F_{p_{n-1},p_n}(x)), g(F_{p_n,p_{n+1}}(x)), g(F_{p_n,p_n}(x))\})$$

i.e. $g(F_{p_n,p_{n+1}}(x)) \le \varphi(\max\{g(F_{p_{n-1},p_n}(x)), g(F_{p_n,p_{n+1}}(x))\})$.

If $g(F_{p_{n-1},p_n}(x)) \le g(F_{p_n,p_{n+1}}(x))$, then by Lemma 3.1, $\lim_{n\to\infty} g(F_{p_n,p_{n+1}}(x)) = 0.$

Again, if $g(F_{p_{n-1},p_n}(x)) \ge g(F_{p_n,p_{n+1}}(x))$ then $g(F_{p_n,p_{n+1}}(x)) \le \varphi(g(F_{p_{n-1},p_n}(x)))$, hence by Lemma 3.1, $\lim_{n\to\infty} g(F_{p_n,p_{n+1}}(x)) = 0.$

Next we claim that $\{P_n\}$ is a Cauchy sequence, otherwise, if $\{P_n\}$ is not Cauchy Sequence, then by 1.1 $\exists\, \varepsilon_0 > 0$, $t_0 > 0$ and sets of positive integer $\{m_i\}, \{n_i\}$ such that,

(i) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$,

and (ii) $g(F_{p_{m_i},p_{n_i}}(t_0)) > g(1-\varepsilon_0)$ and $g(F_{p_{m_{i-1}},p_{n_i}}(t_0) \le g(1-\varepsilon_0)$.

Now,

$$g(F_{p_{m_i},p_{n_i}}(t_0)) \le g(F_{p_{m_i},p_{m_i-1}}(t_0)) + g(F_{p_{m_i-1},p_{n_i}}(t_0))$$

i.e. $g(F_{p_{m_i},p_{n_i}}(t_0)) \le g(1-\varepsilon_0) + g(F_{p_{m_i},p_{m_i-1}}(t_0))$, as $i \to \infty$

or $\lim_{i\to\infty} g(F_{p_{m_i},p_{n_i}}(t_0)) = g(1-\varepsilon_0)$.

Also,

$g(1-\varepsilon_0) < g(F_{p_{m_i},p_{n_i}}(t_0)) \le g(F_{p_{n_i},p_{n_i+1}}(t_0)) + g(F_{p_{n_i+1},p_{m_i}}(t_0))$, hence

$$g(1-\varepsilon_0) = \lim_{i\to\infty} g(F_{p_{n_i+1},p_{m_i}}(t_0)),$$

i.e. $g(F_{p_{n_i+1},p_{m_i}}(t_0)) = g(F_{fp_{n_i},fp_{m_i-1}}(t_0))$

$$\leq \varphi(\max\{g(F_{p_{n_i},p_{m_i-1}}(t_0)), g(F_{p_{n_i+1},p_{n_i}}(t_0)), g(F_{p_{m_i},p_{m_i-1}}(t_0)), g(F_{p_{m_i-1},p_{n_i+1}}(t_0))\})$$

or $g(F_{p_{n_i+1},p_{m_i}}(t_0)) \leq \varphi(\max\{g(F_{p_{n_i},p_{m_i-1}}(t_0)), g(F_{p_{n_i+1},p_{n_i}}(t_0)), g(F_{p_{m_i},p_{m_i-1}}(t_0)), (g(F_{p_{m_i-1},p_{n_i}}(t_0)) + g(F_{p_{n_i+1},p_{n_i}}(t_0)))\})$.

Hence $g(1-\varepsilon_0) \leq \varphi(\max\{g(1-\varepsilon_0), g(1-\varepsilon_0)\})$ (by taking limit)

i.e. $g(1-\varepsilon_0) \leq \varphi(g(1-\varepsilon_0))$ i.e. $g(1-\varepsilon_0) < g(1-\varepsilon_0)$.

This is not possible. Hence $\{P_n\}$ is Cauchy sequence. Since (*X*, *F*; *t*) is complete, $\{p_n\} \to z \in X$. We claim that $z = fz$, otherwise, if $z \neq fz$, then

$$g(F_{fz,p_{n+1}}(x)) = g(F_{fz,fp_n}(x)) \leq \varphi(\max\{g(F_{z,p_n}(x)), g(F_{z,fz}(x)), g(F_{p_n,p_{n+1}}(x)), g(F_{p_n,fz}(x))\}),$$

i.e. $g(F_{fz,z}(x)) \leq \varphi(\{g(F_{z,z}(x)), g(F_{z,fz}(x)), g(F_{z,z}(x)), g(F_{z,fz}(x))\})$ (by taking limit as $n \to \infty$)

i.e. $g(F_{fz,z}(x)) \leq \varphi(g(F_{z,fz}(x)) < g(F_{z,fz}(x))$.

This is not possible, hence $z = fz$, i.e. z is a fixed point of *f*. For uniqueness suppose z, z' are fixed points of *f*. Then

$$g(F_{z,z'}(x)) = g(F_{fz,fz'}(x)) \leq \varphi(\{g(F_{z,z'}(x)), g(F_{z,z}(x)), g(F_{z',z'}(x)), g(F_{z',z}(x))\})$$

i.e. $g(F_{z,z'}(x)) \leq \varphi(g(F_{z,z'}(x))) < g(F_{z,z'}(x))$, which is possible only if $z = z'$.

This proves the uniqueness.

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