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HYPERGRAPHS AND FERMATS THEOREM

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ABSTRACT

Graphs have been used to prove fundamental results in areas of number theory. In this paper we study the proof of Fermat's theorem and Euler's theorem using Hypergraphs.

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Key words: *Hypergraphs, Fermat's theorem, Euler's theorem.*

1. INTRODUCTION

Graph theory is becoming increasingly significant as it is applied to other areas of mathematics, science and technology. It is being actively used in fields as varied as biochemistry (genomics), electrical engineering (communication networks and coding theory), computer science(algorithms and computation) and operations research(scheduling).

Prime numbers have fascinated Mathematicians since the ancient Greeks. Euclid has given the first proof for existence of infinity of primes 2, 3, 5, 7, 11, 13,….Many Mathematicians were trying to find out the pattern for the same. Prime numbers have remained the main object of study for many years. Some of the important theorems or results related to prime numbers are the Fundamental theorem of Arithmetic, Fermat's theorem, Wilson's theorem, Perfect number conjectures, Prime Number theorem, Goldbach Conjecture and so on. Proof for Prime Factorisation theorem was discussed in [6]. Bichitra Kalita has discussed the graph theoretic proof of Goldbach Conjecture.

Fermat's theorem is an important property of integers to a prime modulus.There are many proofs for Fermat's Little Theorem. The first known proof was communicated by Euler in his letter dated 6th March 1742 to Goldbach. The idea of the graph theoretic proof together with some number theoretic results, was used to prove Euler's generalization to non-prime modulus.

2. DEFINITIONS

2.1 Graph: A graph $G(V,E)$ consists of a finite non empty set *V* called as set of vertices together with a set *E* called the set of edges of undirected pair of distinct points of V.

2.2 Simple graph: A graph $G(V,E)$ is said to be *simple* if it has no self loops and parallel edges.

2.3 Multi graph: A graph $G(V,E)$ is said to be a *multi-graph* if it is not a simple graph.

2.3 Degree of a vertex: The number of edges incident with the vertex with self loops counted twice is called as the *degree* of that vertex. A vertex of a graph is called a *pendent vertex* if the degree of the vertex is one.

2.4 Complete graph: A graph $G(V,E)$ is said to be *complete* if there is an edge between every pair of vertices. A complete graph is also called as the *universal graph or a clique*. The degree of every vertex in a complete graph is $n-1$ and it has $\frac{n(n-1)}{2}$ edges. The complete graph of n vertices is denoted by K_n.

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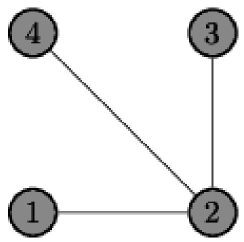
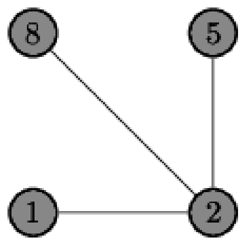
2.5 Bipartite graph: A graph $G(V,E)$ is said to be *bipartite graph* if the vertex set V is partitioned into two disjoint subsets say V_1 and V_2 consisting of m and n vertices whose edge has one vertex in V_1 and other vertex in V_2. It is said to be *complete bipartite* if every vertex of V_1 is incident with every other vertex of V_2. It is denoted by $K_{m,n}$. The bipartite graph $K_{1,n}$ is called as the *star* whose vertex set consist of 2 vertex sets V_1 and V_2 consisting of only one vertex and n vertices respectively where one vertex of V_1 is incident with all the n vertices of V_2 .

2.6 Weighted graph: A weighted graph associates a label (weight) with every edge in the graph. Weights are usually real numbers. They may be restricted to rational numbers or integers.

2.7 Equivalent graphs: Two graphs are said to be equivalent if the vertices can be relabeled to make them equal.





Hypergraph

Hypergraphs, a generalisation of graphs have been widely and deeply studied in Berge (1973, 1984, 1989) and quite often have proved to be a successful tool to represent and model concepts and structures in various areas of Computer Science and Discrete Mathematics. In Mathematics, a hypergraph is a generalisation of a graph in which an edge can connect any number of vertices. A hypergraph *H* is a pair *H=(X, E)* where *X* is a set of elements called as nodes or vertices and *E* is a set of non-empty subsets of *X* called hyperedges or edges. Therefore *E* is a subset of P(X) -{0} where P(X) is the power set of *X*. While graph edges are pairs of nodes, hyperedges are arbitrary sets of edges and can therefore contain an arbitrary number of nodes. Hypergraphs where all hyperedges have the same cardinality generally called as *k* uniform hypergraph-- a hypergraph where all the hyperedges have size k.

2.8 A **Backward hyperarc** or simply **B-arc** is a hyperarc $E=(T(E),H(E))$ with $|H(E)|=1$.

2.9 A **Forward hyperarc** or simply **F-arc** is a hyperarc $E=(T(E),H(E))$ with $|T(E)|=1$.

2.10 A ***B-graph* (or B-hypergraph)** is a hypergraph whose hyperarcs are B-arcs.

A ***F-graph* (or F-hypergraph**) is a hypergraph whose hyperarcs are F-arcs.

A ***BF-graph* (or BF-hypergraph)** is a hypergraph whose hyperarcs are either B-arcs or F-arcs.

3. NEW DEFINITIONS

3.1 Prime vertex hypergraph: If the vertices of a hypergraph are prime numbers then it is called as the prime vertex hypergraph.

3.2 Weighted hypergraph: The weight of the hypergraph is defined with respect to vertices as well as edges. The weight at the head node of the B graph is defined as the product of the prime vertex edges merging at the node. The weight of the edge deleted hypergraph at the head node is defined as the sum of the prime vertices of the B graph.

3.2 Complete hypergraph: A hypergraph *G* (*V*, *E*) is said to be *complete* if there is an edge between every pair of vertices.

3. PROOF OF FERMAT'S THEOREM USING HYPERGRAPHS

3.1. Fermat's Theorem: For any prime *p* and any *a* in Z such that *a* is not congruent to zero mod p,
$a^{p-1} \equiv 1 (\text{mod } p).$

Proof: Consider the B graph whose hyperarcs are of weight a and let there be p number of arcs, where p is a prime number. Let the weight of the B graph at the head node be defined as the product of the prime vertices that is a^p. Delete any edge from the B graph. Let the weight of edge deleted hypergraph be defined as sum of the prime vertices of the hypergraph that is a multiple of a the weight of the edge deleted B graph divides the weight of the edge deleted graph. Hence the theorem.

3.2. Euler's theorem: For m ≥ 2 in Z^+ and any $a \in Z$, such that $(a, m) = 1,\ a^{\phi(m)} \equiv 1(\bmod m)$, where $\phi(m)$ is the number of invertible integers modulo m.

Proof: Construct a hypergraph with vertices as natural numbers 1, 2, 3,…. Let the initial vertex be a_1=1 and the end vertex be some positive integer a_m=m greater than or equal to 2. That is the vertex set is {a_1, a_2, a_3,……}. Consider all possible edge sets {a_1, a_2},{a_1, a_3},{a_1. a_4},…..{a_1, a_2, a_3},{a_1, a_3, a_4},{a_1, a_2,. a_4},{a_1, a_2, a_3, a_4},{a_1, a_2, a_3, a_5}},…………Let the weight of the hypergraph be defined as the sum of the vertices.

If m = p is prime, all non-zero integers modulo p are invertible, so $\phi(p) = p-1$ and Euler's theorem becomes Fermat's Little theorem. Euler's function $\phi(p) = p-1$ is the number of vertices from the initial vertex to the vertex p-1. That is $\phi(p)$ is the vertex set {1, 2, 3,… p-1}.

If m is not prime then the Euler's function is defined as the edge set consisting of relatively prime vertices whose weight is a multiple of m.

Example 3.2.1: Let m be the prime p = 5. The vertex set is {1, 2, 3, 4}. Then the Euler function $\phi(5) = \{1,2,3,4\}$.

Example 3.2.2: Let m be the composite number 6. The vertex set is {1, 2, 4, 5}. Then the edge set whose weight is a multiple of 6 is {1, 5, 2, 4}. Euler function $\phi(6)$ is the set {1, 5}, which consists of only co-primes.

Example 3.2.3: Let m be the prime p=11. The vertex set is {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Then the Euler function is same as the vertex set. That is $\phi(11) = \{1,2,3,4,5,6,7,8,9,10\}$.

Example 3.2.4: Let m be the composite number 8. Then the set of edges whose weight is 8 are {1, 7},{2, 6},{3, 5}. Euler function $\phi(8)$ is the union of the edge sets {1, 7}, {3, 5} comprising only co-primes, that is the edge set {1, 3, 5, 7} whose sum is a multiple of 8. Euler's function is $\phi(8)$ =4.

Example 3.2.5: Let m be the composite number 15. Then the set of edges whose weight is 15 are {1, 14}, {2, 13}, {4, 11}, {7, 8}. Euler's function $\phi(15)$ is the union of the above edge sets {1, 14}, {2, 13}, {4, 11}, {7, 8} comprising co-primes that is the edge set {1, 2, 4, 7, 8, 11, 13, 14} whose sum is a multiple of 15. Euler's function is $\phi(15)$ =8.

Keith Conrad made some changes in Fermat's theorem to get Euler theorem. [4] We will see the proof of the same using graphs.

Consider two equivalent hypergraphs G_1 and G_2. Let the vertices of G_1 be named 1, 2, 3,…..p-1. Let the vertices of G_2 be 1.a, 2.a, 3.a,……..(p-1)a. Let the weights of G_1 and G_2 at the end vertex be defined as the product of the vertices. The weights of G_1 and G_2 at the end vertex a_{p-1} are respectively the products 1.2.3….p-1 and a.2a.3a………(p-1)a. Since G_1 and G_2 are equivalent, we have their weights also to be equivalent. That is

$$\{1.2.3.......p-1\} \equiv \{a.2a.3a.......(p-1)a\}(\bmod p)$$

$$\Rightarrow 1 \equiv a^{p-1}(\bmod p)$$

4. CONCLUSION

Thus we have discussed the proof of Fermat's theorem and Euler's theorem using hypergraphs. We can also explore the proofs of number theory theorems using graphs.

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