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NOTE ON A THEOREM OF ANKENY AND RIVLIN

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ABSTRACT

If $p(z)$ is a polynomial of degree n and does not vanish in $|z| < 1$, then it was shown by Dewan, Hans and Kaur [7] [Journal of Interdisciplinary Mathematics, Vol (13), No 2, (2010), pp. 163-166] that

$$\{M(p,R)\}^s \le \left(\frac{R^{ns}+1}{2}\right)\{M(p,1)\}^s$$

In this paper ,we obtain a generalization of the above inequality , which in turn also generalize as well as improve the result due to N.C. Ankeny and T. J. Rivlin [1] [Pacific. J. Math., vol. 5 (1955), pp. 849-852].

***Key Words**: Polynomial, Inequality, Derivatives, Zeros.*

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1. INTRODUCTION

For an arbitrary entire function $f(z)$, let $M(f,r) = \max_{|z|=r} |f(z)|$ and $m(f,k) = \min_{|z|=k} |f(z)|$ .Then for a polynomial $p(z)$ of a degree n , it is a simple consequence of a maximum modulus principle (for ref. See [4, vol 1, p, 137.prob III, 269]) that

$$M(p\,,R) \le R^n\, M\,(p\,,1\,) \quad \text{for} \quad R \ge 1. \tag{1.1}$$

The result is best possible and equality holds for $p(z) = \alpha\, z^n$ where $|\alpha| = 1$.
While concerning the estimate of $|p^/(z)|$ in terms of $|p(z)|$ on $|z| = 1$ for the class of polynomials having no zeros in $|z| < 1$, it was conjectured by P. Erdös and later by lax [3] that if $p(z) \ne 0$ in $|z| < 1$, then

$$M\left(p^/\,,1\right) \le \frac{n}{2}\, M\,(p\,,1\,) \tag{1.2}$$

The result is best possible and equality holds for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.
For the class of polynomials $p(z)$ of degree n not vanishing in $|z| < k$, $k \ge 1$, Malik [5] proved

$$\max_{|z|=1} |p^/(z)| \le \frac{n}{1+k} \max_{|z|=1} |p(z)| \tag{1.3}$$

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The result is best possible and equality holds for $p(z) = (z+k)^n$.

Govil [6] improved the inequality (1.3) and proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k}\left\{\max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)|\right\} \tag{1.4}$$

The inequality (1.4) is best possible and equality holds for $p(z) = (z+k)^n$.

It was shown by Ankeny and Rivlin [1] that if $p(z) \neq 0$ in $|z| < 1$, then inequality (1.1) can be replaced by a sharper inequality.

Theorem: A If $p(z)$ is a polynomial of degree n, which does not vanish in $|z| < 1$, then

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right) M(p,1), \quad R \geq 1 \tag{1.5}$$

Recently Dewan et al [7] proved the following generalization as well as an improvement of Theorem A.

Theorem: B If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then for every positive integer s

$$\{M(p,R)\}^s \leq \left(\frac{R^{ns}+1}{2}\right)\{M(p,1)\}^s, \quad R \geq 1 \tag{1.6}$$

In this paper we generalize Theorem B, which in turn also generalize as well as improve Theorem A. More precisely, we prove

Theorem: 1 If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \geq 1$, then for every positive integer s, we have

$$\{M(p,R)\}^s \leq \left(\frac{R^{ns}+k}{1+k}\right)\{M(p,1)\}^s, \quad R \geq 1 \tag{1.7}$$

Remark: 1 For $s = 1$ and $k = 1$ the above result reduces to inequality (1.5).

If we take $k = 1$ in Theorem 1, then our result reduces to Dewan, Hans and Kaur [7].

Theorem: 2 If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \geq 1$, then for every positive integer s, we have

$$\{M(p,R)\}^s \leq \left(\frac{R^{ns}+k}{1+k}\right)\{M(p,1)\}^s - \left(\frac{R^{ns}-k}{1+k}\right)\{M(p,1)\}^{s-1} m(p,k), \quad R \geq 1 \tag{1.8}$$

If we take $k = 1$ in Theorem 2, then we get the following result due to Dewan, Hans and Kaur [7].

Corollary: If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \geq 1$, then for every positive integer s, we have

$$\{M(p,R)\}^s \leq \left(\frac{R^{ns}+1}{2}\right)\{M(p,1)\}^s - \left(\frac{R^{ns}-1}{2}\right)\{M(p,1)\}^{s-1} m(p,1), \quad R \geq 1 \tag{1.9}$$

The result is best possible and equality holds for $p(z) = z^n + k^n$.

2. PROOFS OF THE THEOREMS

Proof of Theorem: 1 Let $M(p,1)=\max_{|z|=1}|p(z)|$. Since $p(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, therefore, by inequality (1.3), we have

$$|p'(z)|\le\frac{n}{1+k}\,M(p,1),\quad \text{for } |z|=1$$

Now $p'(z)$ is a polynomial of degree $n-1$, therefore, it follows by (1.1) that for all $r\ge 1$, and $0\le\theta<2\pi$,

$$|p'(re^{i\theta})|\le\frac{n}{1+k}\,r^{n-1}\,M(p,1) \tag{2.1}$$

Also for each θ, $0\le\theta<2\pi$ and $R\ge 1$, we have

$$\{p(Re^{i\theta})\}^s-\{p(e^{i\theta})\}^s=\int_1^R\frac{d}{dt}\{p(te^{i\theta})\}^s\,dt$$
$$=\int_1^R s\{p(te^{i\theta})\}^{s-1}\,p'(te^{i\theta})e^{i\theta}\,dt$$

This imples

$$|\{p(Re^{i\theta})\}^s-\{p(e^{i\theta})\}^s|\le s\int_1^R\left|p(te^{i\theta})^{s-1}\right||p'(te^{i\theta})|\,dt,$$

Which on combining with inequality (1.1) and (2.1) , we get

$$|\{p(Re^{i\theta})\}^s-\{p(e^{i\theta})\}^s|\le\frac{ns}{1+k}\int_1^R t^{ns-1}\{M(p,1)\}^s\,dt,$$
$$=\left(\frac{R^{ns}-1}{1+k}\right)\{M(p,1)\}^s.$$

Which implies

$$|p(Re^{i\theta})|^s\le|p(e^{i\theta})|^s+\left(\frac{R^{ns}-1}{1+k}\right)\{M(p,1)\}^s$$
$$\le\{M(p,1)\}^s+\left(\frac{R^{ns}-1}{1+k}\right)\{M(p,1)\}^s \tag{2.2}$$

Hence from (2.2) we conclude that

$$\{M(p,R)\}^s\le\left(\frac{R^{ns}+k}{1+k}\right)\{M(p,1)\}^s.$$

This completes the proof of Theorem 1.

Proof of Theorem: 2 The proof of Theorem 2 follows on the same lines as that of Theorem 1 by using inequality (1.4) instead of (1.3) . But for the sake of completeness we give a brief outline of the proof. Since $p(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, therefore, by inequality (1.4), we have

$$\max_{|z|=1}|p'(z)|\le\frac{n}{1+k}\{M(p,1)-m(p,k)\}\quad\text{for } |z|=1$$

Now $p'(z)$ is a polynomial of degree $n-1$, therefore, it follows by (1.1) that for all $r\ge 1$, and $0\le\theta<2\pi$,

$$|p'(re^{i\theta})|\le\frac{n}{1+k}\,r^{n-1}\{M(p,1)-m(p,k)\} \tag{2.3}$$

Also for each θ, $0 \le \theta < 2\pi$ and $R \ge 1$, we have

$$\left|\{p(Re^{i\theta})\}^{s} - \{p(e^{i\theta})\}^{s}\right| \le s \int_{1}^{R} \left|p(te^{i\theta})^{s-1}\right| \left|p^{/}(te^{i\theta})\right| dt$$

Which on combining with inequality (1.1) and (1.2) , we get

$$\left|\{p(Re^{i\theta})\}^{s} - \{p(e^{i\theta})\}^{s}\right| \le \left(\frac{R^{ns}-k}{1+k}\right)\{M(p,1)\}^{s}\{M(p,1) - m(p,k)\}$$

Which implies

$$\left|p(Re^{i\theta})\right|^{s} \le \{M(p,1)\}^{s} + \left(\frac{R^{ns}-k}{1+k}\right)\{M(p,1)\}^{s}\{M(p,1) - m(p,k)\}$$

From which the proof of Theorem 2 follows.

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