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EXISTENCE RESULTS FOR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

V. KRISHNAVENI*
Department of Mathematics,
Sri Ramakrishna Mission Vidyalaya, College of Arts and Science, Coimbatore, India.

K. SATHIYANATHAN
Department of Mathematics,
Sri Ramakrishna Mission Vidyalaya, College of Arts and Science, Coimbatore, India.

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ABSTRACT

In this paper, we investigate the existence results for nonlinear impulsive fractional differential equations with integral boundary conditions by using fixed point theorem and Green's function.

Keywords: *Integral boundary; impulsive; fractional differential equation; Green's function; fixed point theorem.*

1. INTRODUCTION

For the last decades, fractional calculus has received a great attention because fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various processes of science and engineering. The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. For an introduction of the basic theory of impulsive differential equations, see Lakshmikantham *et al.*[11], Bainov and Simeonov [4], and Samoilenko and Perestyuk [14] and the references therein. Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so forth. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers [1, 2, 5, 6, 9, 10, 15-17, 19] and the references therein.

The Greens functions for boundary-value problems for ordinary and fractional differential equations have been investigated in detail in [8, 20]

In [3], the authors have studied the existence of solutions for fractional integro- differential equations with impulsive and integral conditions. In [7] the authors have studied a class of nonlinear fractional differential equations with impulsive and fractional integral boundary conditions.

In [18] the authors investigated the expression and properties of Green's function for a second-order singular boundary value problem with integral boundary conditions and delayed argument.

In [12] the authors discussed nonlinear impulsive fractional differential equations involving the two-point and integral boundary conditions. The authors in [13] proved the existence and uniqueness of solutions of differential equations of fractional order with integral boundary conditions.

Inspired by the above works, we consider the existence of solutions for impulsive fractional differential equations with integral boundary conditions

$$\begin{cases} {}^{c}D_{0+}^{q}y(t) = \omega(t)f\left(t, y(t), y'(t)\right), \ 1 < q < 2, \ t \in J = [0,1], \ t \neq t_k, \\ \Delta y|_{t=t_k} = I_k\left(y(t_k)\right), \ \Delta y'|_{t=t_k} = J_k\left(y(t_k)\right), \ t_k \in (0,1), \ k = 1,2,\dots,n, \\ y(0) = y(1) = \int_0^1 g(s)y(s)ds, \end{cases} \tag{1.1}$$

Corresponding Author: K. Sathiyanathan, Department of Mathematics, Sri Ramakrishna Mission Vidyalaya, College of Arts and Science, Coimbatore, India.

where ${}^{C}D_{0_+}^{q}$is the Caputo fractional derivative, $\omega(t): C(J, R_+)$, $J = [0,1]$, $f: C(J \times R \times R, R)$, $I_k, J_k: C(R, R)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^+)$ and $y(t_k^-)$ represent the right-hand limit of $y(t)$ at $t = t_k$ respectively, $k = 1,2,3, \dots, n$, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$. $\Delta y'|_{t=t_k}$ has a similar meaning for $y'(t)$, and $g \in L^1[0,1]$ is nonnegative.

The rest of this paper is organized as follows. In section 2, we give some preliminaries about the fractional integral and derivative operator. In section 3, we present the expression and properties of Green's function associated with problem (1.1). In section 4, we get some existence results for problem (1.1) by means of standard fixed point theorem. Finally, in section 5, the example is also illustrated.

2. PRELIMINARIES

In this section, we give some preliminaries for discussing the solvability of problem (1.1).

Definition 2.1: The fractional (arbitrary) order integral of the function $h \in L^1(J, R_+)$ of order $q \in R_+$ is defined by
$$I_{0^+}^{q} h(t) = \frac{1}{\Gamma(q)} \int_0^1 (t-s)^{q-1} h(s) ds,$$
where $\Gamma(.)$ is the Euler Gamma Function.

Definition 2.2: For a function h given on the interval J, the Caputo-type fractional derivative of order $q > 0$ is defined by
$${}^{C}D_{0^+}^{q} h(t) = \frac{1}{\Gamma(n-q)} \int_0^1 (t-s)^{n-q-1} h^n(s) ds, \quad n = [q] + 1,$$
where the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.1: Let > 0, then the differential equation ${}^{C}D^{q} h(t) = 0$ has the following solution:
$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{(n-1)} t^{n-1}, \qquad c_i \in R, i = 0,1,2, \dots, n-1, n = [q] + 1.$$

Lemma 2.2: Let $q > 0$, then $I^{q}\,{}^{C}D_{0^+}^{q} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{(n-1)} t^{n-1}$, for some $c_i \in R$, $i = 0,1,2, \dots, n-1, n = [q] + 1$.

We define the set of functions as follows

Let $J' = [0,1] \setminus \{t_1, t_2, \dots\dots, t_n\}$ and
$$PC[J, R] = \{y: J \to R;\ y \in C\big((t_k, t_{k+1}), R\big), y(t_k^+)\ and\ y(\,t_k^-)\ exist\ \ with\ y(\,t_k^-) = y(t_k), k = 1,2,3, \dots, n\}$$
$$PC^1[J, R] = \{y' \in PC[J, R];\ y'(t_k^+)\ and\ y'(\,t_k^-)\ exist\ \ and\ y'\ is\ left\ continuous\ at\ t_k, k = 1,2,3, \dots, n\}$$

Then $PC[J, R]$ is a Banach space with the norm
$\|y\|_{PC} = \sup_{t \in J} |y(t)|$, $PC^1[J, R]$ is a Banach space with the norm
$\|y\|_{PC^1} = \max\{\|y\|_{PC}, \ \|y'\|_{PC}\}$.

3. EXPRESSION AND PROPERTIES OF GREEN'S FUNCTION

Definition 3.1: A function $y \in PC^1[J, R] \cap C^2[J', R]$ with its Caputo derivative of order q existing on J is a solution of problem (1.1) if it satisfies (1.1). We give the following hypotheses:
(H_1) $\omega : J \to [0, +\infty)$ is a continuous function, and there exist $t_0 \in J$ such that $\omega(t_0) > 0$;
(H_2) $f : J \times R \times R \to R$ is a continuous functions.
(H_3) $I_k, J_k : R \to R$ are continuous functions.
(H_4) $g \in L^1[0,1]$ is nonnegative and $\mu \in [0,1)$,
where $\mu = \int_0^1 g(t) dt$ (3.1)

Consider the following fractional impulsive boundary value problem :
$$\begin{cases} {}^{C}D_{0_+}^{q} y(t) = \sigma(t), 1 < q < 2, \ t \in J_1 = J \setminus \{t_1, t_2, \dots, t_n\}, \\ \Delta y|_{t=t_k} = I_k\big(y(t_k)\big), \ \Delta y'|_{t=t_k} = J_k\big(y(t_k)\big), \ t_k \in (0,1), \ k = 1,2, \dots, n, \\ y(0) = y(1) = \int_0^1 g(s) y(s) ds, \end{cases} \tag{3.2}$$

Proposition 3.1: The solution of problem (3.2) can be expressed by

$$y(t) = \sum_{i=1}^{n+1} H_2(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)} \sigma(s)ds \; - \sum_{i=1}^{n} H_1(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds + \sum_{i=1}^{n} H_2(t,t_i) I_i\big(y(t_i)\big)$$
$$- \sum_{i=1}^{n} H_1(t,t_i) J_i\big(y(t_i)\big), \qquad t\epsilon\ (t_k, t_{k+1}), \tag{3.3}$$
$$k = 0,1,2, \dots n, t_0 = 0, t_{n+1} = 1.$$

where

$$H_1(t,s) = G_1(t,s) + \frac{1}{1-\mu} \int_0^1 g(u) G_1\,(s,u)du \tag{3.4}$$

$$H_2(t,s) = G_1^{'}(t,s) + \frac{1}{1-\mu} \int_0^1 g(u) G_1^{'}\,(s,u)du \tag{3.5}$$

$$G_1(t,s) = \begin{cases} t(1-s), & t \le s, \\ s(1-t), & t \ge s, \end{cases} \tag{3.6}$$

$$G_1^{'}(t,s) = \begin{cases} -t \;\; or \;\;\; (1-s), & t \le s, \\ (1-t) \;\; or \;\; (-s), & t \ge s, \end{cases} \tag{3.7}$$

Proof: suppose that y is a solution of (3.2). Then, for some constants $b_0, b_1 \in R$, we have

$$y(t) = I_{0^+}^{q} \sigma(t) - b_0 - b_1 t$$
$$= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - b_0 - b_1 t, \;\; t \in [0,1].$$
$$y^{'}(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds - b_1.$$

If $t \in (t_1, t_2]$, then for some constants $c_0,\; c_1 \in R$ we can write

$$y(t) = \int_{t_1}^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - c_0 - c_1(t - t_1),$$
$$y^{'}(t) = \int_{t_1}^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - c_1.$$

Using impulse conditions $\Delta y|_{t=t_1} = I_1\big(y(t_1)\big), \quad \Delta y^{'}|_{t=t_1} = J_1\big(y(t_1)\big),$

$$-c_0 = \int_0^{t_1} \frac{(t_1 - s)^{q-1}}{\Gamma(q)} \sigma(s)ds - b_0 - b_1 t_1 + I_1\big(y(t_1)\big),$$
$$-c_1 = \int_0^{t_1} \frac{(t_1 - s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds - b_1 + J_1\big(y(t_1)\big).$$

Thus

$$y(t) = \int_{t_1}^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + \int_0^{t_1} \left(\frac{(t_1 - s)^{q-2}(t - t_1)}{\Gamma(q-1)} + \frac{(t_1 - s)^{q-1}}{\Gamma(q)} \right) \sigma(s)ds - b_o - b_1 t$$
$$+ J_1\big(y(t_1)\big)(t - t_1) + I_1(y(t_1))$$

$$y^{'}(t) = \int_{t_1}^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds \;\; + \int_0^{t_1} \left(\frac{(t_1 - s)^{q-2}}{\Gamma(q-1)} \right) \sigma(s)ds \; - b_1 + J_1\big(y(t_1)\big).$$

If $\in (t_k, t_{k+1}]$, repeating the above procedure, we obtain

$$y(t) = \int_{t_k}^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\frac{(t_i - s)^{q-2}(t - t_i)}{\Gamma(q-1)} \; + \; \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right) \sigma(s)ds - b_0$$
$$- b_1 t + \sum_{i=1}^{k} J_i\big(y(t_i)\big)(t - t_i) \;\; + \;\; \sum_{i=1}^{k} I_i\big(y(t_i)\big) \tag{3.10}$$

By the boundary conditions, we have

$$b_0 = -\int_0^1 g(t)y(t)dt \tag{3.11}$$

$$b_1 = \int_{t_k}^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\frac{(t_i-s)^{q-2}(1-t_i)}{\Gamma(q-1)} + \frac{(t_i-s)^{q-1}}{\Gamma(q)}\right)\sigma(s)ds + \sum_{i=1}^{k} J_i\big(y(t_i)\big)(t-t_i) + \sum_{i=1}^{k} I_i\big(y(t_i)\big) \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10), we have

$$y(t) = \sum_{i=1}^{n+1} G_1'(t,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \sum_{i=1}^{n} G_1(t,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds + \sum_{i=1}^{n} G_1'(t,t_i)I_i\big(y(t_i)\big) - \sum_{i=1}^{n} G_1(t,t_i)J_i\big(y(t_i)\big) + \int_0^1 g(t)y(t)dt$$

where

$$\int_0^1 g(t)\,y(t)dt = \int_0^1 g(t)\sum_{i=1}^{n+1} G_1'(t,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\sigma(s)ds\,dt - \int_0^1 g(t)\sum_{i=1}^{n} G_1(t,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\sigma(s)dsdt - \int_0^1 g(t)\sum_{i=1}^{n} G_1(t,t_i)J_i\big(y(t_i)\big)dt + \int_0^1 g(t)\sum_{i=1}^{n} G_1'(t,t_i)I_i\big(y(t_i)\big)\,dt + \int_0^1 g(t)dt \times \int_0^1 g(t)y(t)dt$$

Therefore, we have

$$\int_0^1 g(s)\,y(s)ds = \frac{1}{1-\int_0^1 g(s)ds}\Bigg[\int_0^1 g(u)\sum_{i=1}^{n+1} G_1'(s,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\sigma(s)du\,ds - \int_0^1 g(u)\sum_{i=1}^{n} G_1(s,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\sigma(s)duds - \int_0^1 g(s)\sum_{i=1}^{n} G_1(s,t_i)J_i\big(y(t_i)\big)ds + \int_0^1 g(s)\sum_{i=1}^{n} G_1'(s,t_i)I_i\big(y(t_i)\big)\,ds\Bigg]$$

and

$$y(t) = \sum_{i=1}^{n+1} G_1'(t,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \sum_{i=1}^{n} G_1(t,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds + \sum_{i=1}^{n} G_1'(t,t_i)I_i\big(y(t_i)\big) - \sum_{i=1}^{n} G_1(t,t_i)J_i\big(y(t_i)\big)\frac{1}{1-\mu}\Bigg[\int_0^1 g(u)\sum_{i=1}^{n+1} G_1'(s,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\sigma(s)du\,ds - \int_0^1 g(u)\sum_{i=1}^{n} G_1(s,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\sigma(s)duds - \int_0^1 g(s)\sum_{i=1}^{n} G_1(s,t_i)J_i\big(y(t_i)\big)ds + \int_0^1 g(s)\sum_{i=1}^{n} G_1'(s,t_i)I_i\big(y(t_i)\big)\,ds\Bigg]$$

Let $\quad H_1(t,s) = G_1(t,s) + \dfrac{1}{1-\mu}\displaystyle\int_0^1 g(u)G_1(s,u)du$

$$H_2(t,s) = G_1'(t,s) + \frac{1}{1-\mu}\int_0^1 g(u)G_1'(s,u)du$$

Then,

$$y(t)=\sum_{i=1}^{n+1}H_2(t,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\sigma(s)ds-\sum_{i=1}^{n}H_1(t,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds+\sum_{i=1}^{n}H_2(t,t_i).I_i\big(y(t_i)\big)$$
$$-\sum_{i=1}^{n}H_1(t,t_i)\,.J_i\big(y(t_i)\big)$$

Proposition 3.2: If (H_4) holds, then we have

$$H_1(t,s)>0, G_1(t,s)>0, \quad for\ \ t,s\in(0,1) \tag{3.13}$$
$$H_1(t,s)\geq 0, G_1(t,s)\geq 0, \quad for \quad t,s\in(0,1) \tag{3.14}$$

Proof: From the definition of $H_1(t,s)$ and $G_1(t,s)$, it is easy to obtain the result of (3.13) and (3.14).

Proposition 3.3: For $t,s\in[0,1]$, we have

$$e(t)e(s)\leq G_1(t,s)\leq G_1(s,s)=s(1-s)=e(s)\leq\bar{e}=\max_{t\in[0,1]}e(s)=\tfrac{1}{4} \tag{3.15}$$

Proof: In fact, for $t\in J$, $s\in(0,1)$, we have

Case-1: If $0<t\leq s<1$, then

$$\frac{G_1(t,s)}{G_1(s,s)}=\frac{t(1-s)}{s(1-s)}=\frac{t}{s}\leq 1.$$

Case-2: If $0<s\leq t<1$, then

$$\frac{G_1(t,s)}{G_1(s,s)}=\frac{s(1-t)}{s(1-s)}=\frac{1-t}{1-s}\leq\frac{1-s}{1-s}\leq 1.$$

In addition, by the definition of $G_1(t,s)$, it is easy to obtain that

$$G_1(t,s)\leq G_1(t,s),\qquad \forall t\in J,\qquad s\in(0,1).$$

Therefore

$$G_1(t,s)\leq G_1(t,s)=e(s),\forall t,s\in J.$$

Similarly, we can prove that

$$G_1(t,s)\geq e(t)e(s).$$

In fact, for all t,s we have that

Case-1: If $t\leq s$, then

$$\frac{G_1(t,s)}{G_1(s,s)}=\frac{t(1-s)}{s(1-s)}=\frac{t}{s}\geq t\geq t(1-t).$$

Case-2: If $t\leq s$, then

$$\frac{G_1(t,s)}{G_1(s,s)}=\frac{s(1-t)}{s(1-s)}=\frac{1-t}{1-s}\geq(1-t)=t(1-t).$$

So we have

$$G_1(t,s)\geq e(t)e(s),\quad \forall\ t,s\in J.$$

Proposition 3.4: If (H_4) holds, then for $t,s\in[0,1]$, we have

$$\rho e(s)\leq H_1(t,s)\leq\gamma s(1-s)=\gamma e(s)\leq\frac{1}{4}\gamma \tag{3.16}$$

where

$$\gamma=\frac{1}{1-\mu},\qquad \rho=\frac{\int_0^1 e(\tau)g(\tau)d\tau}{1-\mu} \tag{3.17}$$

Proof: By (3.4) and (3.14), we have

$$H_1(t,s)=G_1(t,s)+\frac{1}{1-\mu}\int_0^1 G_1(s,\tau)g(\tau)d\tau$$

$$\geq \frac{1}{1-\mu}\int_0^1 G_1(s,\tau)g(\tau)d\tau$$
$$\geq \frac{\int_0^1 e(\tau)g(\tau)d\tau}{1-\mu} s(1-s)$$
$$= \rho\, e(s), t \in [0,1].$$

On the other hand, noticing $G_1(t,s) \leq s(1-s)$, we obtain

$$H_1(t,s) = G_1(t,s) + \frac{1}{1-\mu}\int_0^1 G_1(s,\tau)g(\tau)d\tau$$
$$= s(1-s) + \frac{1}{1-\mu}\int_0^1 s(1-s)g(\tau)d\tau$$
$$= s(1-s)\left[1 + \frac{1}{1-\mu}\int_0^1 g(\tau)d\tau\right]$$
$$= s(1-s)\left[1 + \frac{1}{1-\mu}\right]$$
$$= \gamma e(s), \qquad t \in [0,1].$$

The proof of Proposition 3.4 is complete.

Proposition 3.5: If (H_4) holds, then for $t, s \in [0,1]$, we have

$$H_2(t,s) \leq \gamma(1-t) \leq \gamma \tag{3.18}$$

Proof:

$$H_2(t,s) = G_1^{'}(t,s) + \frac{1}{1-\mu}\int_0^1 G_1^{'}(s,\tau)g(\tau)d\tau$$
$$= 1 + \frac{1}{1-\mu}\int_0^1 g(\tau)d\tau$$
$$= \frac{1}{1-\mu}$$
$$\leq \gamma$$

It follows from Proposition (3.1) that:

Lemma 3.1: If $(H_1) - (H_2)$ hold, then a function $y \in PC^1[J,R] \cap C^2[J^{'},R]$ is a solution of problem (1.1) iff $y \in PC^1[J,R]$ is a solution of the impulsive fractional integral equation

$$\begin{aligned} y(t) = &\sum_{i=1}^{n+1} H_2(t,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}[\omega(s)f(s,y(s),y^{'}(s))]ds \\ &- \sum_{i=1}^{n} H_1(t,t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}[\omega(s)f(s,y(s),y^{'}(s))]ds + \sum_{i=1}^{n} H_2(t,t_i)I_i\big(y(t_i)\big) \\ &- \sum_{i=1}^{n} H_1(t,t_i)J_i\big(y(t_i)\big) \end{aligned}$$

$$k = 0,1,2,\ldots\ldots,n,\ t_0 = 0, t_{n+1} = 1 \tag{3.19}$$

Proof: Let $y \in PC^1[J,R] \cap C^2[J^{'},R]$ be a solution of the boundary value problem (1.1), then by the same method as used in Proposition 3.1, we can prove that y is a solution of the fractional integral equation (3.19).

Conversely, assume that y satisfies the impulsive fractional integral equation (3.19).

If $t \in [0,t_1]$ then $y(0) = \int_0^1 g(s)y(s)ds,\ y(1) = \int_0^1 g(s)y(s)ds$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α we get

$${}^CD^\alpha y(t) = h(t) \text{ for each } t \in [0,t_1].$$

If $t \in [t_k, t_{k+1}), k = 1,2, \dots, n$ and using the fact that ${}^{C}D^{\alpha}C = 0$, where C is a constant, we get
${}^{C}D^{\alpha}y(t) = h(t)$ for each $t \in [t_k, t_{k+1})$.

Also we can easily show that

$$\Delta y|_{t=t_k} = I_k\big(y(t_k)\big),\ \Delta y'|_{t=t_k} = I_k\big(y(t_k)\big),\ k = 1,2, \dots \dots \dots \dots, n.$$

Hence the proof.

Define $T:\ PC^1[J,R] \to PC^1[J,R]\ by$

$$\begin{aligned} Ty(t) = \sum_{i=1}^{n+1} H_2(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)}[\omega(s)f(s,y(s),y'(s))]ds \\ - \sum_{i=1}^{n} H_1(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)}[\omega(s)f(s,y(s),y'(s))]ds + \sum_{i=1}^{n} H_2(t,t_i)I_i\big(y(t_i)\big) \\ - \sum_{i=1}^{n} H_1(t,t_i)J_i\big(y(t_i)\big), \quad k = 0,1,2, \dots n, t_0 = 0, t_{n+1} = 1 \end{aligned} \tag{3.20}$$

Using Lemma (3.1), Problem (1.1) reduces to a fixed point problem $x = Tx$, where T is given by (3.20). Thus, The problem (1.1) has a solution iff operator T has a fixed point.

Lemma 3.2: Assume that $(H_1) - (H_4)$ hold. Then $T:\ PC^1[J,R] \to PC^1[J,R]$ is completely continuous.

Proof: Note that the continuity of $f, g, \omega, I_k\ and\ J_k$ together with $H_1(t,s)\ and\ H_2(t,s)$ ensures the continuity of T.

Let $\Omega \subset PC^1[J,R]$ be bounded. Then there exists positive constants λ_1, λ_2 and λ_3, such that
$|f(t,y(t),y'(t)| \le \lambda_1, |I_k(y)| \le, \lambda_2 and, |J_k(y)| \le \lambda_3\ \forall y \in \Omega.\ Thus\ \forall\ y \in \Omega$, we have

$$\begin{aligned} |(Ty)(t)| &\le \sum_{i=1}^{n+1} |H_2(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \left|\omega(s)f\big(s,y(s),y'(s)\big)\right| ds \\ &\quad + \sum_{i=1}^{n} |H_1(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \left|\omega(s)f\big(s,y(s),y'(s)\big)\right| ds + \sum_{i=1}^{n} |H_2(t,t_i)|\left|I_i\big(y(t_i)\big)\right| \\ &\quad + \sum_{i=1}^{n} |H_1(t,t_i)| \left|J_i\big(y(t_i)\big)\right| \\ &\le \sum_{i=1}^{n+1} |H_2(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)}[c\lambda_1]ds + \sum_{i=1}^{n} |H_1(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)}[c\lambda_1]ds + \lambda_2 \sum_{i=1}^{n} |H_2(t,t_i)| \\ &\quad + \lambda_3 \sum_{i=1}^{n} |H_1(t,t_i)| \\ &\le \frac{\gamma c(n+1)\lambda_1}{\Gamma(q+1)} + \frac{n\gamma c\lambda_1}{4\Gamma(q)} + n\gamma\lambda_2 + \frac{n\gamma\lambda_3}{4} = \lambda \end{aligned} \tag{3.21}$$

where

$$c = \max_{t \in J} \omega(t) \tag{3.22}$$

Furthermore, for any $t \in (t_k, t_{k+1}]$

$$\begin{aligned} \left|(Ty)'(t)\right| &\le \sum_{i=1}^{n+1} \left|G''_{1st}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \left[\left|\omega(s)f\big(s,y(s),y'(s)\big)\right|\right] ds \\ &\quad + \sum_{i=1}^{n} |G'_{1t}(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \left[\left|\omega(s)f\big(s,y(s),y'(s)\big)\right|\right] ds + \sum_{i=1}^{n} \left|G''_{1st}(t,t_i)\right|\left|I_i\big(y(t_i)\big)\right| \\ &\quad + \sum_{i=1}^{n} |G'_{1t}(t,t_i)| \left|J_i\big(y(t_i)\big)\right| \end{aligned}$$

$$\leq \sum_{i=1}^{n+1} \left|G''_{1st}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)}[c\lambda_1]ds + \sum_{i=1}^{n} |G'_{1t}(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)}[c\lambda_1]ds + \lambda_2 \sum_{i=1}^{n} \left|G''_{1st}(t,t_i)\right| + \lambda_3 \sum_{i=1}^{n} |G'_{1t}(t,t_i)|$$

$$\leq \frac{c(n+1)\lambda_1}{\Gamma(q+1)} + \frac{nc\lambda_1}{\Gamma(q)} + n\lambda_2 + n\lambda_3 = \bar{\lambda} \tag{3.23}$$

On the other hand, From (3.23) for $t_1,\ t_2 \,\epsilon\, (t_k,\ t_{k+1}]$ with $t_1 < t_2$, we have

$$\left|(Ty)'(t_2) - (Ty)'(t_1)\right| \leq \int_{t_1}^{t_2} |(Ty)'(s)|\, ds \leq \bar{\lambda}(t_2 - t_1) \tag{3.24}$$

It follows from (3.21), (3.23) and (3.24) that T is equicontinuous on all subintervals $(t_k, t_{k+1}], k = 1,2 \dots n.$ Thus by Arzela-Ascoli Theorem, the operator $T:\ PC^1[J,R] \to PC^1[J,R]$ is completely continuous.

To prove our main result, we also need the following lemma.

Lemma 3.3 (Schauder Fixed point Theorem): Let D be non empty, closed, bounded, convex subset of a banach space X, and suppose that $T: D \to D$ is completely continuous operator. Then T has fixed point $x \in D$.

4. EXISTENCE OF SOLUTIONS

In this section we apply Lemma 3.3 to establish the existence of solutions to problem (1.1). Let us define

$$\theta = \lim_{|y|+|z|\to\infty} \left(\max_{t\in J} \frac{|f(t,y,z)|}{|y|+|z|} \right),$$

$$\theta_1 = \lim_{|y|\to\infty} \frac{|I_k(y)|}{|y|}, \qquad \theta_2 = \lim_{|y|\to\infty} \frac{|J_k(y)|}{|y|}, \qquad k = 1,2,\dots\dots n$$

Theorem 4.1: Assume that $(H_1) - (H_4)$ hold. Suppose further that

$$\delta = \max\{\delta_1, \delta_2\} < 1 \tag{4.1}$$

where

$$\delta_1 = \frac{2c(n+1)\gamma\theta}{\Gamma(q+1)} + \frac{2cn\gamma\theta}{4\Gamma(q)} + n\gamma\theta_1 + \frac{n\gamma\theta_2}{4},$$

and

$$\delta_2 = \frac{2c(n+1)\theta}{\Gamma(q+1)} + \frac{2cn\theta}{\Gamma(q)} + n\theta_1 + n\theta_2$$

where c is defined in (3.22) . Then problem (1.1) has atleast one solution $x \in PC^1[J,R] \cap C^2[J',R]$.

Proof: We shall use Schauder's fixed point theorem to prove that T has a fixed point. First, recall that the operator $T:\ PC^1[J,R] \to PC^1[J,R]$ is completely continuous (see the proof of lemma (3.2)).

On account of (4.1), we can choose $\theta' > \theta,\ \theta_1' > \theta_1\ and\ \theta_2' > \theta_2$ such that

$$\delta'_1 = \frac{2c(n+1)\gamma\theta'}{\Gamma(q+1)} + \frac{2cn\gamma\theta'}{4\Gamma(q)} + n\gamma\theta_1' + \frac{n\gamma\theta_2'}{4} \tag{4.2}$$

and

$$\delta'_2 = \frac{2c(n+1)\theta'}{\Gamma(q+1)} + \frac{2cn\theta'}{\Gamma(q)} + n\theta_1' + n\theta_2' \tag{4.3}$$

By the definition of θ, there exists $l > 0$ such that

$$\begin{aligned} &|f(t,y,z)| < \theta'(|y|+|z|), \quad \forall\, t \in J,\ \ |y|+|z| > l, \\ &|f(t,y,z)| < \theta'(|y|+|z|) + M_1, \quad \forall\, t \in J,\ \ y,z \in R, \end{aligned} \tag{4.4}$$

where

$$M_1 = \max_{t\in J,\, |y|+|z|\leq l} |f(t,y,z)| < +\infty$$

Similarly, we have

$$\begin{cases} |I_k(y)| < \theta_1'|x| + M_k, & \forall\, y \in R, \qquad k = 1,2 \dots n \\ |J_k(y)| < \theta_2'|x| + \bar{M}_k, & \forall\, y \in R, \qquad k = 1,2 \dots n \end{cases} \tag{4.5}$$

Where $M_1, M_2, M_k, \bar{M}_k$ are positive constants.

It follows from (3.20), (4.4) and (4.5) that

$$\begin{aligned}
|(Ty)(t)| \quad &\le \sum_{i=1}^{n+1} |H_2(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \left[\left|\omega(s) f\big(s, y(s), z\,(s)\big)\right|\right] ds \\
&\quad + \sum_{i=1}^{n} |H_1(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \left[\left|\omega(s) f\big(s, y(s), z(s)\big)\right|\right] ds \\
&\quad + \sum_{i=1}^{n} |H_2(t,t_i)| \left|I_i\big(y(t_i)\big)\right| + \sum_{i=1}^{n} |H_1(t,t_i)| \left|J_i\big(y(t_i)\big)\right| \\
&\le \sum_{i=1}^{n+1} |H_2(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} [\gamma(\theta'(|y|+|z|)+M)] ds \\
&\quad + \sum_{i=1}^{n} |H_1(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} [\gamma(\theta'(|y|+|z|)+M)] ds \\
&\quad + \sum_{i=1}^{n} |H_2(t,t_i)| (\theta_1'|y| + M_k) + \sum_{i=1}^{n} |H_1(t,t_i)| \,(\theta_2'|y| + \bar{M}_k) \\
&\le \frac{(n+1)c\gamma}{\Gamma(q+1)} \left[\big(2\theta'(\|y\|_{PC^1}) + M\big) + \frac{nc\gamma}{4\Gamma(q)} \left[\big(2\theta'(\|y\|_{PC^1}) + M\big)\right] \right] \\
&\quad + n\gamma\big(\theta_1'(\|y\|_{PC^1}) + M_k\big) + n\frac{\gamma}{4}\big(\theta_2'(\|y\|_{PC^1}) + \bar{M}_k\big) \\
&\le \delta_1' \|y\|_{PC^1} + M^{(1)}
\end{aligned} \tag{4.6}$$

Where δ_1' is defined by (4.2) and $M^{(1)}$ is defined by

$$M^{(1)} = \frac{M(n+1)c\gamma}{\Gamma(q+1)} + \frac{Mnc\gamma}{4\Gamma(q)} + n\gamma M_k + n\frac{\gamma}{4}\bar{M}_k$$

Similarly, from (3.20), (4.4) and (4.5), we get

$$\begin{aligned}
\left|(Ty)\,'(t)\right| &\le \sum_{i=1}^{n+1} \left|G''_{1st}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \left[\left|\omega(s) f\big(s, y(s), z\,(s)\big)\right|\right] ds \\
&\quad + \sum_{i=1}^{n} \left|G'_{1t}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \left[\left|\omega(s) f\big(s, y(s), z(s)\big)\right|\right] ds \\
&\quad + \sum_{i=1}^{n} \left|G''_{1st}(t,t_i)\right| \left|I_i\big(y(t_i)\big)\right| + \sum_{i=1}^{n} \left|G'_{1t}(t,t_i)\right| \left|J_i\big(y(t_i)\big)\right| \\
&\le \sum_{i=1}^{n+1} \left|G''_{1st}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} [\gamma(\theta'(|y|+|z|)+M)] ds \\
&\quad + \sum_{i=1}^{n} \left|G'_{1t}(t,t_i)\right| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} [\gamma(\theta'(|y|+|z|)+M)] ds \\
&\quad + \sum_{i=1}^{n} \left|G''_{1st}(t,t_i)\right| (\theta_1'|y| + M_k) + \sum_{i=1}^{n} \left|G'_{1t}(t,t_i)\right| (\theta_2'|y| + \bar{M}_k) \\
&\le \frac{(n+1)c}{\Gamma(q+1)} \left[\big(2\theta'(\|y\|_{PC^1}) + M\big) + \frac{nc}{\Gamma(q)} \left[\big(2\theta'(\|y\|_{PC^1}) + M\big)\right] \right] + n\big(\theta_1'(\|y\|_{PC^1}) + M_k\big) \\
&\quad + n\big(\theta_2'(\|y\|_{PC^1}) + \bar{M}_k\big) \\
&\le \delta_2' \|y\|_{PC^1} + M^{(2)}
\end{aligned} \tag{4.7}$$

Where δ_2' is defined by (4.3) and $M^{(2)}$ is defined by

$$M^{(2)} = \frac{M(n+1)c}{\Gamma(q+1)} + \frac{Mnc}{\Gamma(q)} + nM_k + n\bar{M}_k$$

It follows from (4.6) and (4.7) that

$$\|Ty\|_{PC^1} \le \delta' \|Ty\|_{PC^1} + M', \forall\, y \in PC^1[J, R]$$

where

$$\delta' = \max\{\delta_1', \delta_2'\} < 1, \qquad M' = \max\{M^{(1)}, M^{(2)}\}.$$

Hence, we can choose a sufficiently large $r > 0$ such that $T(B_r) \subset B_r$, where $B_r = \{y \in PC^1 : \|y\|_{PC^1} \le r$

Consequently, Lemma 3.3 implies that T has a fixed point in B_r, and the proof is complete.

Remark 4.1: Condition (4.1) is certainly satisfied if $\frac{|f(t,y,z)|}{|y|+|z|} \to 0$ uniformly in $t \in J$ as $|y| + |z| \to +\infty$, $\frac{|I_k(y)|}{|y|} \to 0$ as $|y| \to +\infty$, $\frac{|J_k(y)|}{|y|} \to 0$ as $|y| \to +\infty$, $(k = 1,2 \dots n)$.

5. EXAMPLE

We consider the following boundary value problem:

$$\begin{cases} {}^{C}D_{0+}^{\frac{3}{2}} x(t) = b_0\, t^{\frac{1}{6}} \left[b_1 \sqrt[5]{b_2 t - x + x'} - \frac{1}{20} x' - b_3 I_n (1 + x^2) \right], & t \in J, \quad t \neq \frac{1}{2}, \\ \Delta x|_{t_1 = \frac{1}{2}} = \frac{1}{10}\, x\left(\frac{1}{2}\right), \quad \Delta x'|_{t_1 = \frac{1}{2}} = \frac{1}{6}\, x\left(\frac{1}{2}\right), \\ x(0) = x(1) = \int_0^1 \frac{t^2}{25} x(t) d, \end{cases} \tag{5.1}$$

Here $q = \frac{3}{2}, n = 1, b_0, b_1, b_2, b_3$ are positive real numbers.

This problem has at least one solution in

$PC^1[J, R] \cap C$^2 $[J', R]$ Where $J' = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$.

Proof: It follows from (5.1) that

$$\omega(t) = b_0\, t^{\frac{1}{6}},$$

$$f(t, x, x') = b_1 \sqrt[5]{b_2 t - x + x'} - \frac{1}{20} x' - b_3 I_n (1 + x^2) \tag{5.2}$$

$$t_1 = \frac{1}{2}, \qquad I_1(x) = \frac{1}{10} x, \qquad J_1(x) = \frac{1}{6}\, x \tag{5.3}$$

From the definition of ω, f, I_1 and J_1, it is easy to see that $(H_1) - (H_4)$ hold. So

$\theta \leq \frac{1}{20}$, $\theta_1 \leq \frac{1}{10}, \theta_2 \leq \frac{1}{6}$, $\mu = \int_0^1 \frac{t^2}{25} dt$. We have

$$G_1(t, s) = -\begin{cases} s(1 - t), 0 \leq s \leq t \leq 1, \\ t(1 - s), 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_t'(t, s) = \begin{cases} s \qquad, 0 \leq s \leq t \leq 1, \\ (s - 1), 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_{1s}'(t, s) = \begin{cases} (t - 1), 0 \leq s \leq t \leq 1, \\ t \qquad, 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_{1st}''(t, s) = 1$$

$$\mu \leq \frac{1}{25}, \quad H_1(t, s) \leq \frac{25}{96}, \qquad H_2(t, s) \leq \frac{25}{24},$$

$$\delta_1 \leq 0.25, \qquad \delta_2 \leq 0.4.$$

Thus, our conclusion follows from theorem 4.1.

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