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FIXED POINT THEOREM IN CONE b_2-METRIC SPACE OVER BANACH ALGEBRA

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ABSTRACT

The concept of cone b_2-metric space is introduced as a generalization of cone 2-metric space and b_2-metric space. We proved Hardy and Roger's type fixed point theorem in cone b_2-metric space over banach algebra. Our results generalize many well known results in fixed point theory.

Mathematics Subject Classification: *47H10.*

Keywords: *fixed point theorems, cone metric spaces, banach algebra, spectral radius.*

1. INTRODUCTION

Czerwik introduced the concept of b-metric space as a generalization of metric spaces [12]. Many fixed point theorems were proved by several authors using b-metric [13]. The concept of 2-metric space was introduced by Gahler in 1960's keeping in view the area of triangle as an example [16]. Naidu and Prasad showed that every convergent sequence in a 2-metric space need not be a Cauchy sequence. Different authors proved that there is no relation between 2-metric and ordinary metric and that the contraction mappings in ordinary metric spaces and 2-metric spaces are unrelated. For fixed point theorems in 2-metric spaces [1], [14], [15], [16]. In 2007, Huang and Zang introduced the concept of cone metric space by generalizing ordinary metric space [7]. Several authors proved many fixed and common fixed point theorems in cone metric spaces. Liu and Xu proved fixed point theorems in cone metric spaces over banach algebra. For fixed point theorems in cone metric spaces, cone b-metric spaces. See [3], [5], [8], [9], [11], [18], [19]. In 2012, B.Singh *et.al*, introduced cone 2-metric space by generalizing 2-metric and cone metric and proved some fixed point theorems. For some results on cone 2-metric space, see [2], [6], [17]. Recently, Zead Mustafa *et. al*, generalized the above 2-metric and b-metric spaces and introduced $\boldsymbol{b_2}$-metric space [10]. They proved some fixed point theorems in partially ordered $\boldsymbol{b_2}$-metric space.

In this work, we generalized $\boldsymbol{b_2}$-metric space and cone metric space as cone $\boldsymbol{b_2}$-metric space giving example for the existance of such spaces. We then proved a fixed point theorem using Hardy and Roger's [4] type contractive condition over banach algebra. An example is given supporting our main theorem.

2. PRELIMINARY NOTES

Let A always represents a real Banach Algebra. Then we have the following properties: For all $x, y, z \in X$ and $a \in \mathbb{R}$:

1. $(xy)z = x(yz)$
2. $x(y+z) = xy + xz$ and $(x+y)z = xy + xz$
3. $a(xy) = (ax)y = x(ay)$
4. $\|xy\| \le \|x\|\|y\|$

Throughout this paper, we shall assume that the Banach Algebra has a unit (multiplicative identity) e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

Preposition 2.1 [8]: Let A be a real Banach algebra with a unit e and $x \in A$. If the spectral radius $r(x)$ of x is less than 1. $i.e., r(x) = \lim_{x\to a} \|x^n\|^{\frac{1}{n}} < 1$, then $e - x$ is invertible and $(e-x)^{-1} = \sum_{i=0}^{\infty} x^i$.

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Remark 2.2 [18] For all $x \in A$, we have $r(x) \le \|x\|$.

Definition 2.3 [18] Let A be a Banach algebra and P be a subset of E. P is called a cone if:
1. P is closed, nonempty and $P \neq \{\theta\}$.
2. $a, b \in R, a, b \ge 0, x, y \in P \Rightarrow ax + by \in P$.
3. $x \in P$ and $-x \in P \Rightarrow x = \theta$

For a cone $P \subset A$, define a partial ordering $\preccurlyeq$ w.r.t P as follows:
1. $x \preccurlyeq y$ iff $y - x \in P$.
2. $x \prec y$ will stand for $x \preccurlyeq y$ and $x \neq y$.
3. $x \ll y$ means $y - x \in intP$, where $intP$ denotes interior of P.

If $int\ P \neq \emptyset$, then P is called a solid cone.

The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in A$, we have
$$\theta \preccurlyeq x \preccurlyeq y \Longrightarrow \|x\| \le k\|y\|.$$

The least positive number satisfying the above is called the normal constant of P. Throughout this paper, we shall assume that P is a solid cone.

Definition 2.4: Let X be a non-empty set and $s \ge 1$. Suppose the mapping $d: X \times X \times X \to A$ satisfies:
1. for every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq \theta$.
2. $\theta \preccurlyeq d(x, y, z)$, for all $x, y, z \in X$ and $d(x, y, z) = \theta$ if and only if at least two of x, y, z are equal.
3. $d(x, y, z) = d(p(x, y, z))$ for all $x, y, z \in X$ and for all permutations $p(x, y, z)$ of x, y, z.
4. $d(x, y, z) \preccurlyeq s[d(x, y, w) + d(x, w, z) + d(w, y, z)]$ for all permutations $x, y, z, w \in X$.

Then d is called a cone $\boldsymbol{b_2}$-metric on X, and (X, d) is called a cone $\boldsymbol{b_2}$-metric space.

Example 2.5: Let $X = \mathbb{R}, A = \mathbb{R}^2$ and $\|(x, y)\| = |x| + |y|$ for all $(x, y) \in A$. Let $P = \{(x, y): x, y \ge 0\}$ be a normal cone. Define $d_1: X^2 \to A$ as $d_1(\alpha, \beta) = (\|\alpha - \beta\|^p, l\|\alpha - \beta\|^p)$ for all $\alpha, \beta \in A$ where $l \ge 0$ and $p > 1$ are constants.

Now define $d: X^3 \to A$ as $d(x, y, z) = d_1(\alpha, \beta)$ where $\alpha, \beta \in X$ are such that $\|\alpha - \beta\| = \min\{\|x - y\|, \|y - z\|, \|z - x\|\}$.

Then it follows from [12] that (X, d) is a cone $\boldsymbol{b_2}$-metric space with $s = 3^{p-1}$ but not a cone 2-metric space.

Definition 2.6 Let $\{x_n\}$ be a sequence in cone $\boldsymbol{b_2}$-metric space over banach algebra (X, d). Then
1. $\{x_n\}$ is said to be $\boldsymbol{b_2}$-convergent to $x \in X$ if $\lim_{n\to\infty} d(x_n, x, a) = \theta$ for all $a \in X$.
2. $\{x_n\}$ is said to be $\boldsymbol{b_2}$-Cauchy sequence in X if $\lim_{n\to\infty} d(x_n, x_m, a) = \theta$ for all $x \in X$.
3. (X, d) is said to be $\boldsymbol{b_2}$-complete if every $\boldsymbol{b_2}$-Cauchy sequence is a $\boldsymbol{b_2}$-convergent in X.

Definition 2.7 [18] Let P be a solid cone in a Banach algebra A. A sequence $\{u_n\} \subseteq P$ is a c –sequence if for each c with $\theta \ll c$ there exist $m \in \mathbb{N}$ such that $u_n \ll c$ for all $n \ge m$.

Let (X, d) be a cone metric space over banach algebra. The following properties are often used (particularly when dealing with non-normal cone).

(p_1) If $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \ll z$.
(p_2) If $\theta \preccurlyeq x \ll c$ for each $c \in int\ P$, then $x = \theta$.
(p_3) If $x \preccurlyeq y + c$ for each $c \in int\ P$, then $x \preccurlyeq y$.
(p_4) If $\theta \preccurlyeq x \preccurlyeq y$ and $\theta \preccurlyeq u$ then $\theta \preccurlyeq ux \preccurlyeq uy$.
(p_5) If $\theta \preccurlyeq x_n \preccurlyeq y_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\theta \preccurlyeq x \preccurlyeq y$.
(p_6) If $\theta \preccurlyeq d(x_n, x) \preccurlyeq b_n$ and $b_n \to \theta$, then $x_n \to x$.
(p_7) If $u \preccurlyeq \lambda u$, where $u \in P$ and $0 < \lambda < 1$, then $u = \theta$.
(p_8) If $c \in int\ P$, $\theta \preccurlyeq u_n$ and $\|u_n\| \to 0$ then there exist an $m \in \mathbb{N}$ such that $u_n \ll c$ for all $n \ge m$.
(p_9) If $\{x_n\}$ and $\{y_n\}$ are c –sequences and $a, b > 0$, then $\{ax_n + by_n\}$ is also a c –sequence.
(p_{10}) If $\{u_n\}$ is a c –sequence in P and $k \in P$ is an arbitrarily given vector, then $\{ku_n\}$ is also a c –sequence.
(p_{11}) If $\{x_n\}$ and $\{y_n\}$ are sequences in A with $x_n \to x$ and $y_n \to y$ as $n \to \infty$, where $x, y \in A$, then we have $x_n y_n \to xy$ as $n \to \infty$.
(p_{12}) If $\{x_n\}$ is a c –sequence in X converging to $x \in X$, then $\{d(x_n, x)\}$ and $\{d(x_n, x_{n+p})\}$ for any $p \in \mathbb{N}$ are also c –sequences.
(p_{13}) If $r(k) < 1$ then $\|k^n\| \to 0$ as $n \to \infty$.
(p_{14}) If $\lambda \in P$ with $r(k) < 1$, then $(e - x)^{-1} \in P$.

(p_{15}) If $x, y \in A$ and x, y commute, then the following holds:
(a) $r(xy) \leq r(x)r(y)$
(b) $r(x+y) \leq r(x) + r(y)$ and
(c) $|r(x) - r(y)| \leq r(x-y)$.
(p_{16}) If $k \in A$ such that $0 \leq r(k) < 1$, then we have
$r((e-x)^{-1}) \leq (1 - r(k))^{-1}$.

3. MAIN RESULTS

Lemma 3.1 Let (X, d) be a complete cone b_2-metric space over banach algebra A with coefficient $s \geq 1$ and P be a solid cone. If $\{x_n\}$ is any sequence in X, then we have

$$d(x_{n+1}, x_n, a) \preccurlyeq kd(x_n, x_{n-1}, a) \Longrightarrow d(x_{n+1}, x_n, x_m) = \theta \quad (3.1)$$

for all $a \in X$ and $m \in \mathbb{N} \cup \{0\}$ where $k \in P$ with $r(k) \in [0, \frac{1}{s})$.

Proof: Since $d(x_{n+1}, x_n, a) \preccurlyeq kd(x_n, x_{n-1}, a)$, repeated application of this to $d(x_{n+1}, x_n, a)$ gives

$$d(x_{n+1}, x_n, a) \preccurlyeq k^n d(x_1, x_0, a). \quad (3.2)$$

If $m < n$, then

$$\begin{aligned} d(x_{n+1}, x_n, x_m) &\preccurlyeq kd(x_n, x_{n-1}, x_m) \\ &\preccurlyeq k^2 d(x_{n-1}, x_{n-2}, x_m) \\ &\dots \dots \dots \dots \\ &\preccurlyeq k^{n-m} d(x_{m+1}, x_m, x_m) \\ &= \theta. \end{aligned}$$

$$\therefore d(x_{n+1}, x_n, x_t) = \theta \text{ for all } m \text{ with } m < n. \quad (3.3)$$

If $m > n$, then

$$\begin{aligned} d(x_{n+1}, x_n, x_m) &\preccurlyeq s[d(x_{n+1}, x_n, x_{m-1}) + d(x_{n+1}, x_{m-1}, x_m) + d(x_{m-1}, x_n, x_m)] \\ &= sd(x_{n+1}, x_n, x_{m-1}) \qquad \text{Using (3.3)} \\ &\preccurlyeq s^2[d(x_{n+1}, x_n, x_{m-2}) + d(x_{n+1}, x_{m-2}, x_{m-1}) + d(x_{m-2}, x_n, x_{m-1})] \\ &= s^2 d(x_{n+1}, x_n, x_{m-2}) \qquad \text{Using (3.3)} \\ &\preccurlyeq s^3 d(x_{n+1}, x_n, x_{m-3}) \\ &\dots. \dots \dots \dots \\ &\preccurlyeq s^{n-m} d(x_{n+1}, x_n, x_n) \\ &= \theta. \end{aligned}$$

$$\therefore d(x_{n+1}, x_n, x_t) = \theta \text{ for all } m \text{ with } m > n. \quad (3.4)$$

Since the case is obvious with $m = n$, from (3.3) and (3.4) we have,

$$d(x_{n+1}, x_n, x_t) = \theta \text{ for all } t \in \mathbb{N} \cup \{0\}.$$

Lemma 3.2 If the hypothesis of Lemma 3.1 hold, then for any $m > n$, we have

$$d(x_n, x_m, a) \preccurlyeq sk^n(e - sk)^{-1} d(x_1, x_0, a). \quad (3.5)$$

Proof For any $m > n$,

$$\begin{aligned} d(x_n, x_m, a) &\preccurlyeq s[d(x_n, x_m, x_{n+1}) + d(x_n, x_{n+1}, a) + d(x_{n+1}, x_m, a)] \\ &= sk^n d(x_1, x_0, a) + sd(x_{n+1}, x_m, a) \qquad \text{Using (3.1) and (3.2)} \\ &\preccurlyeq sk^n d(x_1, x_0, a) + s^2[d(x_{n+1}, x_m, x_{n+2}) + d(x_{n+1}, x_{n+2}, a) + d(x_{n+2}, x_m, a)] \\ &\preccurlyeq sk^n d(x_1, x_0, a) + s^2 k^{n+1} d(x_1, x_0, a) + s^2 d(x_{n+2}, x_m, a) \\ &\dots \dots \dots \dots \\ &\preccurlyeq sk^n d(x_1, x_0, a) + s^2 k^{n+1} d(x_1, x_0, a) + s^3 k^{n+2} d(x_1, x_0, a) \dots \dots \\ &+ s^{m-n-2} k^{m-3} d(x_1, x_0, a) + s^{m-n-2} d(x_{m-2}, x_m, a) \\ &\preccurlyeq (sk^n + s^2 k^{n+1} + s^3 k^{n+2} + \dots + s^{m-n-2} k^{m-3}) d(x_1, x_0, a) \\ &+ s^{m-n-1} k^{m-2} d(x_1, x_0, a) + s^{m-n-1} d(x_{m-1}, x_m, a) \\ &\preccurlyeq (sk^n + s^2 k^{n+1} + s^3 k^{n+2} + \dots + s^{m-n-2} k^{m-3} \\ &+ s^{m-n-1} k^{m-2}) d(x_1, x_0, a) + s^{m-n-1} k^{m-1} d(x_1, x_0, a) \\ &\preccurlyeq (sk^n + s^2 k^{n+1} + \dots + s^{m-n-1} k^{m-2} + s^{m-n} k^{m-1}) d(x_1, x_0, a) \\ &= sk^n (1 + sk + s^2 k^2 + \dots + s^{m-n-1} k^{m-n-1}) d(x_1, x_0, a) \\ &\preccurlyeq sk^n \left(\sum_{i=1}^{\infty} (sk)^i \right) d(x_1, x_0, a) \\ &\preccurlyeq sk^n (e - sk)^{-1} d(x_1, x_0, a). \end{aligned}$$

Theorem 3.3 Let (X, d) be a compelte cone b_2-metric space over banach algebra A with coefficient $s \geq 1$ and P be a solid cone. Let $T: X \to X$ be a mapping satisfying

$$d(Tx, Ty, a) \preccurlyeq k_1 d(x, y, a) + k_2 d(x, Tx, a) + k_3 d(y, Ty, a) + k_4 d(x, Ty, a) + k_5 d(y, Tx, a) \quad (3.6)$$

for all $x, y, a \in X$ where

$$2sr(k_1) + (s+1)r(k_2 + k_3)(s^2 + s)r(k_4 + k_5) < 2. \quad (3.7)$$

Then T has a unique fixed point in X.

Proof Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}\cup\{0\}$.

If $x_n = x_{n+1}$ for some n, then it can be proved that x_n is a fixed point of T. Assume $x_n \neq x_{n+1}$ for all $n \geq 0$.

$$\begin{aligned} d(x_{n+1}, x_n, a) &= d(Tx_n, Tx_{n-1}, a) \\ &\preccurlyeq k_1 d(x_n, x_{n-1}, a) + k_2 d(x_n, x_{n+1}, a) + k_3 d(x_{n-1}, x_n, a) \\ &\quad + k_4 d(x_n, x_n, a) + k_5 d(x_{n-1}, x_{n+1}, a) \\ &\preccurlyeq k_1 d(x_n, x_{n-1}, a) + k_2 d(x_n, x_{n+1}, a) + k_3 d(x_{n-1}, x_n, a) \\ &\quad + k_5 s[d(x_{n-1}, x_{n+1}, x_n) + d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a)] \end{aligned}$$

$i.e.$, $(e - k_2 - sk_5)d(x_{n+1}, x_n, a) \preccurlyeq (k_1 + k_3 + sk_5)d(x_n, x_{n-1}, a)$

$$i.e., \; d(x_{n+1}, x_n, a) \preccurlyeq \alpha d(x_n, x_{n-1}, a) \quad (3.8)$$

where $\alpha = (e - k_2 - sk_5)^{-1}(k_1 + k_3 + sk_5) \in P$.

Because of symmetry of the cone b_2-metric d, we have

$$\begin{aligned} d(x_{n+1}, x_n, a) &= d(x_n, x_{n+1}, a) \\ &= d(Tx_{n-1}, Tx_n, a) \\ &\preccurlyeq k_1 d(x_{n-1}, x_n, a) + k_2 d(x_{n-1}, x_n, a) + k_3 d(x_n, x_{n+1}, a) \\ &\quad + k_4 d(x_{n-1}, x_{n+1}, a) + k_5 d(x_n, x_n, a) \\ &\preccurlyeq k_1 d(x_{n-1}, x_n, a) + k_2 d(x_{n-1}, x_n, a) + k_3 d(x_n, x_{n+1}, a) \\ &\quad + k_4[d(x_{n-1}, x_{n+1}, x_n) + d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a)] \end{aligned}$$

$i.e.$, $(e - k_3 - sk_4)d(x_{n+1}, x_n, a) \preccurlyeq (k_1 + k_2 + sk_4)d(x_n, x_{n-1}, a)$

$$\text{Or} \; d(x_{n+1}, x_n, a) \preccurlyeq \beta d(x_n, x_{n-1}, a) \quad (3.9)$$

where $\beta = (e - k_3 - sk_4)^{-1}(k_1 + k_2 + sk_4) \in P$.

Using (p_{14}) and (p_{15}) we have

$$r(\alpha) \leq \frac{r(k_1) + r(k_3) + sr(k_5)}{1 - r(k_2) - sr(k_5)} \text{ and } r(\beta) \leq \frac{r(k_1) + r(k_3) + sr(k_4)}{1 - r(k_3) - sr(k_4)}$$

If $r(\alpha) > \frac{1}{s}$ and $r(\beta) > \frac{1}{s}$ then we get

$$sr(k_1) + r(k_2) + sr(k_3) + (s^2 + s)r(k_5) > 1 \text{ and}$$
$$sr(k_1) + r(k_2) + sr(k_3) + (s^2 + s)r(k_4) > 1.$$

Upon adding the above two inequalities we get

$$2sr(k_1) + (s+1)r(k_2 + k_3)(s^2 + s)r(k_4 + k_5) > 2$$

which is a contradiction to the hypothesis (3.7). Thus we have either $r(\alpha) < \frac{1}{s}$ or $r(\beta) < \frac{1}{s}$ or both. Take

$$k = \begin{cases} \alpha & if \quad r(\alpha) < \frac{1}{s} \\ \beta & if \quad r(\beta) < \frac{1}{s} \\ \alpha \text{ or } \beta & if \quad both\ are\ true. \end{cases}$$

Then $d(x_{n+1}, x_n, a) \preccurlyeq kd(x_n, x_{n-1}, a)$ for all $n \geq 0$ where $k \in P$ with $r(k) \in \left[0, \frac{1}{s}\right)$.

Keeping in view Lemma (3.1) and (3.2), for $m > n$ we have,

$$d(x_n, x_m, a) \preccurlyeq sk^n(e - sk)^{-1}d(x_1, x_0, a).$$

Using (p_8) and (p_{13}), it can be seen that $\{x_n\}$ is a b_2-Cauchy sequence in X. Since X is b_2-complete, there is an $x \in X$ satisfying $x_n \to x$ as $n \to \infty$.

Now we see that this x is a fixed point of T. For that considering $d(Tx,x,a)$ with triangle inequality and (3.6), we have

$$\begin{aligned} d(Tx,x,a) &\preccurlyeq s[d(Tx,x,Tx_n)+d(Tx,Tx_n,a)+d(Tx_n,x,a)] \\ &\preccurlyeq sd(x_{n+1},x,Tx)+s[k_1d(x,x_n,a)+k_2d(x,Tx,a)+k_3d(x_n,x_{n+1},a) \\ &\quad +k_4d(x,x_{n+1},a)+k_5d(x_n,Tx,a)]+sd(x_{n+1},x,a) \\ &\preccurlyeq sd(x_{n+1},x,Tx)+sk_1d(x,x_n,a)+sk_2d(x,Tx,a)+sk_3d(x_n,x_{n+1},a) \\ &\quad +sk_4d(x,x_{n+1},a)+s^2k_5[d(x_n,Tx,\boldsymbol{x})+d(x_n,x,a)++d(x,Tx,a)+sd(x_{n+1},x,a) \end{aligned}$$

Letting $n\to\infty$, we get $d(Tx,x,a)\preccurlyeq(sk_2+s^2k_5)d(Tx,x,a)$.

Because of Symmetry, we also get $d(Tx,x,a)\preccurlyeq(sk_3+s^2k_4)d(Tx,x,a)$.

Upon adding the above two inequalities, we get

$$d(Tx,x,a)\preccurlyeq[\tfrac{s}{2}(k_2+k_3)+\tfrac{s^2}{2}(k_4+k_5)]d(Tx,x,a) \qquad (3.10)$$

Using (3.7) we have $r\left[\frac{s}{2}(k_2+k_3)+\frac{s^2}{2}(k_4+k_5)\right]<1$

$i.e.$, $d(Tx,x,a)=\theta$ for all $a\in X$.

$i.e.$, $Tx=x$ for all $a\in X$.

$\therefore$ $x$is a fixed point of T.

The Uniqueness can be proved easily by taking $Tx=x$ and $Ty=y$ in (3.6).

Example 3.4: Let $A=\mathbb{R}^2, P=\{(x,y): x,y\geq 0\}$ with $\|(x,y)\|=|x|+|y|$ for all $(x,y)\in A$ and $X=\{(\alpha,0):\alpha\geq 0\}\cup\{(0,2)\}$. Define the 2-metric $d: XxXxX\to A$ as follows:

$$d(A,B,C)=\begin{cases} d(p(A,B,C)) & p \text{ denotes permutation} \\ (1,1) & if\ \ two\ of\ A,B,C\ are\ (0,0),(0,2)\ and \\ & A,B,C\ are\ distinct \\ (\Delta,\Delta) & otherwise. \end{cases}$$

where Δ= square of the area of triangle A,B,C.

Consider the following:

$$d((\alpha,0),(\beta,0),(0,2))\preccurlyeq d((\alpha,0),(\beta,0),(\gamma,0))+d((\alpha,0),(\gamma,0),(0,2))+d((\gamma,0),(\beta,0),(0,2))$$

$i.e.,\ \ (\alpha-\beta)^2\preccurlyeq(\alpha-\gamma)^2+(\gamma-\beta)^2$

which shows that d is not a cone 2-metric, but a cone b_2-metric with $s=2$. Define $T: X\to X$ as

$$T(x,0)=\begin{cases}\left(\frac{x}{2},0\right) & if\ x\geq 10\\ (0,0) & if\ x<10\end{cases} \text{ and } T(0,2)=(0,0).$$

Take $k_1=\left(\frac{1}{4},0\right), k_2=k_3=\left(\frac{1}{18},0\right)$ and $k_4=5=\left(\frac{1}{24},0\right)$.

Clearly, each $k_i(i=1,2,3,4,5)$ and $s=2$ satisfies (3.7) and T satisfies the condition (3.6). Hence, by our main result, T has unique fixed point which is (0,0).

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