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SOME RESULTS ON GENERALIZED SAKAGUCHI TYPE FUNCTIONS

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ABSTRACT

In this paper, we introduce a new subclass $L_s(\alpha,\beta,s,t)$ of analytic function using Sakaguchi type functions. We obtain characterization and subordination results for the functions belonging to these classes. Several interesting consequences of these results are also pointed out.

Keywords: *Sakaguchi type functions, Convolution, Characterization, Subordination.*

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1. INTRODUCTION

Let A be the class of all analytic univalent functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $U = \{z: |z| < 1\}$.

Let $S(\alpha,s,t)$ be the subclass of A consisting of functions given by (1.1) satisfying the condition

$$Re\left\{\frac{(s-t)z\,f'(z)}{f(sz)-f(tz)}\right\} > \alpha, \quad (z\epsilon U, \quad 0 \le \alpha < 1, \ \ t \neq s)$$

We denote by $T(\alpha,s,t)$ the subclass of A consisting of all functions $f(z)$ such that $z\,f'(z) \in S(\alpha,s,t)$. These classes were introduced and studied by Frasin [1]. The class $S(\alpha,1,t)$ was introduced and studied by Owa *et al.* [3] and the class $S(\alpha,1,-1) = S_s(\alpha)$ was introduced and studied by Sakaguchi [4]. Also $S(\alpha,1,0) = S^*(\alpha)$ and $T(\alpha,1,0) = C(\alpha)$, the ususl classes of starlike and convex functions of order α $(0 \le \alpha < 1)$, let K denote the class of functions that are convex in U.

Now we introduce a new subclass $L_s(\alpha,\ \beta,\ s,\ t)$ defined as follows.

Definition 1.1: A function $f(z)\epsilon$ A is said to be in the class $L_s(\alpha,\beta,s,t)$ if it satisfies

$$Re\left\{\frac{(s-t)zf'(z)+\beta(s-t)z^2f''(z)}{(1-\beta)[f(sz)-f(tz)]+\beta z[f(sz)-f(tz)]'}\right\} > \alpha \tag{1.2}$$

Where, $0 \le \alpha < 1$, $0 \le \beta < 1$, $t \neq s$ and $z\epsilon U$.

Remark: Upon setting different values for β, s and t, the class $L_s(\alpha,\beta,s,t)$ reduces to the subclasses $L_s(\alpha,\beta,t)$, $S(\alpha,s,t)$, $T(\alpha,s,t)$, $S(\alpha,t)$, $T(\alpha,t)$, $S(\alpha,1,-1)$, $T(\alpha,1,-1)$, $S^*(\alpha)$, $K(\alpha)$ studied earlier by Frasin[1], Owa *et al.* [3], Sakaguchi[4] and Shilpa and Latha[5].

The purpose of the present paper is to investigate the characterization and subordination results for the functions belonging to the class $L_s(\alpha,\beta,s,t)$.

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2. CHARACTERIZATION RESULTS

In the present section we obtain the Characterization results for the functions in the class $L_s(\alpha,\beta,s,t)$.

Theorem 2.1: If the function $f(z)\epsilon$ A, satisfies the inequality

$$\sum_{n=2}^{\infty}[1+(n-1)\beta][|n-u_n|+(1-\alpha)|u_n|]|a_n|\leq 1-\alpha, \tag{2.1}$$

Where $u_n=\sum_{k=1}^{n}s^{n-k}t^{k-1}$, $0\leq\alpha<1$, $0\leq\beta<1$, $t\neq s$ and $z\epsilon U$, then $f(z)\,\epsilon\, L_s(\alpha,\beta,s,t)$. The result is sharp.

Proof: To prove the theorem it is enough to show that

$$\left|\frac{(s-t)zf'(z)+\beta(s-t)z^2\mathrm{f}''(\mathrm{z})}{(1-\beta)[f(sz)-f(tz)]+\beta z[f(sz)-f(tz)]'}-1\right|<1-\alpha.$$

Since

$$\left|\frac{(s-t)zf'(z)+\beta(s-t)z^2\mathrm{f}''(\mathrm{z})}{(1-\beta)[f(sz)-f(tz)]+\beta z[f(sz)-f(tz)]'}-1\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta](n-u_n)\,a_n\,z^n}{z+\sum_{n=2}^{\infty}[1+(n-1)\beta]u_n\,a_n\,z^n}\right|<\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n|\,|a_n|\,\,|z^{n-1}|}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n|\,|a_n|\,|z^{n-1}|}$$

$$<\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n|\,|a_n|}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n|\,|a_n|}.$$

We see that

$$\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n|\,|a_n|}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n|\,|a_n|}<1-\alpha$$

$$\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n|\,|a_n|\;<(1-\alpha)\left(1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n|\,|a_n|\right)$$

$$\sum_{n=2}^{\infty}[1+(n-1)\beta][|n-u_n|+(1-\alpha)|u_n|]|a_n|\leq 1-\alpha.$$

Remark:

(i) For s=1, we obtain the results in [5].

(ii) For s=1 and β= 0, 1 yield the results due to Owa *et al.* [3].

3. SUBORDINATION RESULTS

In the present section we derive the Subordination results. In order to derive this result we need the following Definitions and Lemma.

Definition 3.1: For two functions f and g in the class A, where f is given by (1.1) and g is given by $\mathrm{g}(z)=z+\sum_{n=2}^{\infty}b_nz^n$, the Hadamard product (or Convolution) $f*g$ is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n=(g*f)(z),\;\;(z\epsilon U).$$

Definition 3.2: Given two functions f and g analytic in U, we say that the function f is subordinate to g in U and write $f\prec g$, if there exists a Schwarz function ω, which is analytic in U, with $\omega(0)=0$ and $|\omega(z)|<1$, such that $f(z)=g\big(\omega(z)\big)$, $z\,\epsilon\,U$.

Definition 3.3: A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by

$$\sum_{n=1}^{\infty}a_nb_nz^n\prec f(z),\quad(z\epsilon U,\;a_1=1).$$

Lemma 3.4: [6] The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$$Re\left\{1+2\sum_{n=1}^{\infty}b_nz^n\right\}>0,\qquad z\,\epsilon U.$$

Theorem 3.5: Let the function $f(z)$ in the class A satisfy the inequality (2.1), and suppose that $g \epsilon K$, then

$$\frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{2[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]}(f*g)(z) \prec g(z) \tag{3.1}$$

Where $z\epsilon U$, $0 \le \alpha < 1$, $0 \le \beta < 1$, $t \neq s$ and

$$Re\{f(z)\} > -\frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)}{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}, z \epsilon U \tag{3.2}$$

The constant factor

$$\frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{2[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]}$$

In the subordination result (3.1) cannot be replaced by a longer one.

Proof: Let the function f defined by (1.1) be in the class $L_s(\alpha,\beta,t)$ and suppose that $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \epsilon K$. Then we have

$$\frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{2[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]}(f*g)(z) \tag{3.3}$$

$$= \frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{2[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]}\left(z + \sum_{n=2}^{\infty} a_n c_n z^n\right)$$

By definition (3.3) the subordination result (3.1) holds true if the sequence

$$\left\{\frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{2[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]} a_n\right\}_{n=1}^{\infty} \tag{3.4}$$

is a subordinating factor sequence with $a_1 = 1$.

In view of Lemma (3.4) it is enough to prove the inequality:

$$Re\left\{1 + 2\sum_{n=1}^{\infty} \frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{2[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]} a_n z^n\right\} > 0, (z \epsilon U) \tag{3.5}$$

Now,

$$Re\left\{1 + \sum_{n=1}^{\infty} \frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]} a_n z^n\right\}$$

$$= Re\left\{1 + \frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]} z + \sum_{n=2}^{\infty} \frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]} a_n z^n\right\}$$

$$\ge 1 - \frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]} r - \sum_{n=2}^{\infty} \frac{[1+(n-1)\beta][|n-u_n|+(1-\alpha)|u_n|]}{[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]} |a_n| r^n$$

$$> 1 - \frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]} r - \frac{(1-\alpha)}{[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]} r > 0$$

Then (3.5) holds in U. This proves the inequality (3.1). The inequality (3.2) follows from (3.1), upon setting

$$g(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \epsilon K$$

To prove the sharpness of the constant

$$\frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{2[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]}$$

We consider the function f_0 defined by

$$f_0(z) = z - \frac{(1-\alpha)}{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]} z^2 \tag{3.6}$$

From (3.1)

$$\frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{2[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]} f_0(z) \prec \frac{z}{1-z} \qquad (3.7)$$

For the function f_0 , it is easy to verify that

$$Min\left\{Re\left\{\frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{2[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]}\, f_0(z)\right\}\right\} = -1/2$$

This shows that the constant

$$\frac{(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]}{2[(1+\beta)[|2-s-t|+(1-\alpha)|s+t|]+(1-\alpha)]}$$

is the best possible, which completes the proof.

Remark: Suitable choices of β and s yield the results due to Frasin and Maslina Darus [2], and Shilpa and Latha [5].

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