International Journal of Mathematical Archive-7(9), 2016, 122-127
Available online through www.ijma.info ISSN 2229 – 5046

MEAN SQUARE SUM LABELING OF PATH RELATED GRAPHS

C. JAYASEKARAN*
Department of Mathematics,
Pioneer Kumaraswamy College, Nagercoil - 629003, Tamil Nadu, India.

M. JASLIN MELBHA
Department of Mathematics,
Women's Christian College, Nagercoil - 629001, Tamil Nadu, India.

(Received On: 29-08-16; Revised & Accepted On: 22-09-16)

ABSTRACT

Let G = (V, E) be a graph. A bijection f: $V(G) \to \{0, 1, \dots, p-1\}$ *is said to be a mean square sum labeling if the induced function* $f^*: E(G) \to N$ *given by* $f^*(uv) = \left\lfloor \frac{[f(u)]^2+[f(v)]^2}{2} \right\rfloor$ *or* $\left\lceil \frac{[f(u)]^2+[f(v)]^2}{2} \right\rceil$ *for every* $uv \in E(G)$ *is injective. A graph which admits a mean square sum labeling is called a mean square sum graph. The concept of mean square sum labeling was introduced by C. Jayasekaran, S. Robinson Chellathurai and M. Jaslin Melbha and they investigated the mean square sum labeling of several standard graphs and some cycle related graphs. In this paper, we prove that composition of paths* P_m *and* P_2, P_n^2, P_n^3, $D_2(P_n)$, *Splitting graph of path* P_n, *Switching of a pendant vertex in path* P_n, T_n, DT_n, *Coconut tree and the graph obtained from* P_n *by attaching* C_3 *in both the end edges of* P_n *are mean square sum graph.*

Key words*: labeling, mean square sum labeling, mean square sum graph.*

1. INTRODUCTION

We begin with simple, finite, connected and undirected graph. For standard terminology and notations we follow Harary [1]. A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a *vertex labeling* (an *edge labeling*). Several types of graph labeling and a detailed survey is available in [2].

S. Somasundaram and R. Ponraj [5] have introduced the notion of mean labeling of graphs. A graph G with p vertices and q edges is called *mean graph* if there is an injective function *f* from the vertices of G to {0, 1, …, q} such that when each edge uv is labeled with $\frac{f(u)+f(v)}{2}$ if f(u)+f(v) is even and with $\frac{f(u)+f(v)+1}{2}$ if f(u)+f(v) is odd, then the resulting edge labels are distinct.

V. Ajitha, S. Arumugam and K. A. Germina [6] have introduced the notion of square sum labeling. A (p, q) graph G is said to be square sum, if there exists a bijection $f : V(G) \to \{0, 1, \dots, p-1\}$ such that the induced function f*: E(G)→N defined by f*(uv) = $[f(u)]^2 + [f(v)]^2$ for every uv ∈ $E(G)$ is injective.

The concept of mean square sum labeling was introduced by C. Jayasekaran, S. Robinson Chellathurai and M. Jaslin Melbha [3] and they investigated the mean square sum labeling of several standard graphs. Not every graph is mean square sum. For example, any complete graph K_n, where n ≥ 6 is not mean square sum. Also they investigated the mean square sum labeling of some cycle related graphs [4]. We are interested to study different classes of graphs, which are mean square sum.

A brief summary of definitions and other information which are necessary for the present investigation are given below.

Corresponding Author: C. Jayasekaran, Department of Mathematics,
Pioneer Kumaraswamy College, Nagercoil - 629003, Tamil Nadu, India.*

Definition 1.1: Let G = $(V(G), E(G))$ be a graph. A bijection f :$V(G)$→{0, 1, ..., p-1} G is said to be a *mean square sum labeling* if the induced function f*: E(G)→N given by f*(uv) = $\left\lfloor\frac{[f(u)]^2+[f(v)]^2}{2}\right\rfloor$ or $\left\lceil\frac{[f(u)]^2+[f(v)]^2}{2}\right\rceil$ for every uv ∈ $E(G)$ is injective.

Definition 1.2: A graph which admits a mean square sum labeling is called a *mean square sum* graph.

Definition 1.3: The *composition* of two graphs G_1 and G_2 denoted by $G=G_1[G_2]$ has vertex set $V(G_1[G_2]) = V(G_1) \times V(G_2)$ and edge set $E(G_1[G_2]) = \{(u_1, v_1)(u_2, v_2) / u_1u_2 \in E(G_1)$ or $u_1 = u_2$ and $v_1v_2 \in E(G_2)\}$.

Definition 1.4: *Square of a graph G* denoted by G^2 has the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance of 1 or 2 apart in G.

Definition 1.5: *Cube of a graph G* denoted by G^3 has the same vertex set as of *G* and two vertices are adjacent in G^3 if they are at a distance at most 3 apart in *G*.

Definition 1.6: For a connected graph *G*, let G' be the copy of *G*. The *shadow graph $D_2(G)$* is obtained by joining each vertex *u* in *G* to the neighbours of the corresponding vertex u' in G'.

Definition 1.7: Let G be a graph. For each point v of a graph G, take a new point v'. Join v' to those points of G adjacent to v . The graph thus obtained is called the *splitting graph* of G. We denote it by *S'(G).*

Definition 1.8: A *vertex switching* of a graph *G* by a vertex of G is the graph G^v which is obtained by removing all the edges incident with *v* and adding all non adjacent edges as edges incident with *v*.

Definition 1.9: A *triangular snake T_n* is obtained from a path $v_1v_2…v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $1 \leq I \leq n - 1$. That is, every edge of the path is replaced by a triangle C_3.

Definition 1.10: A double triangular snake consists of two triangular snakes that have a common path. That is, a *double triangular snake DT_n* is a graph obtained from a path $u_1u_2…u_n$ by joining u_i and u_{i+1} to two new vertices v_i and w_i for $1 \leq I \leq n - 1$.

2. MAIN RESULTS

Theorem 2.1: The composition of paths P_m and P_2 denoted as $P_m[P_2]$ is a mean square sum graph.

Proof: Let $u_1, u_2, …, u_m$ be the vertices of path P_m and v_1, v_2 be the vertices of path P_2. The composition $P_m[P_2]$ consists of 2m vertices, can be partitioned into two sets $V_1 = \{(u_i, v_1) / i = 1, 2, ..., m\}$ and $V_2 = \{(u_i, v_2) / i = 1, 2, ..., m\}$. The edge set is E = {[(u_i, v_1) (u_{i+1}, v_1)], [(u_i, v_2) (u_{i+1}, v_2)], [(u_i, v_1) (u_{i+1}, v_2)], [(u_i, v_2) (u_{i+1}, v_1)], [(u_j, v_1) (u_j, v_2)] / 1≤ j≤ n, 1≤ i≤ n-1}. Let G = $P_m[P_2]$. Define f :V(G)→{0, 1, ..., 2m-1} by f(u_i, v_1) = 2i-2, f(u_i, v_2) = 2i-1, for 1≤ i≤ n. The induced function f*:E(G)→N is defined by f*[(u_i, v_1) (u_{i+1}, v_1)] = $4i^2$-4i+2, f*[(u_i, v_2) (u_{i+1}, v_2)] = $4i^2$+1, f*[(u_i,v_1) (u_{i+1},v_2)] = $4i^2$-2i+2, f*[(u_i,v_2) (u_{i+1},v_1)] = $4i^2$-2i+1, for 1≤i≤n-1 and f*[(u_i,v_1) (u_i,v_2)] = $4i^2$-6i+3, 1≤ i≤ n is injective. Hence $P_m[P_2]$ is a mean square sum graph.

Example 2.2: A mean square sum labeling of $P_7[P_2]$ is given in figure 1.

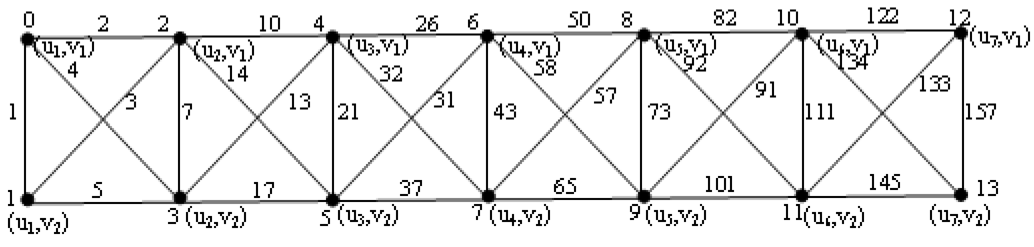

Figure-1: $P_7[P_2]$

Theorem 2.3: P_n^2 is a mean square sum graph.

Proof: Let $u_1u_2…u_n$ be the path P_n. Let G = P_n^2. Then *V(G)* = {u_1, u_2, …, u_n} and *E(G)*= {u_iu_{i+1}, u_ju_{j+2} /1≤ i≤ n-1, 1≤ j≤ n-2}. Clearly, G has n vertices and 2n-3 edges. Define f: V(G)→{0, 1, ..., n-1} as follows f(u_i) = i-1, 1≤ i≤ n. The induced function f*: E(G)→N is defined by f*(u_iu_{i+1}) = i^2-i+1, 1≤ i≤ n-1 and f*(u_iu_{i+2}) = i^2+1, 1≤ i≤ n-2 is injective. Hence P_n^2 is a mean square sum graph.

Example 2.4: A mean square sum labeling of P_7^2 is given in figure 2.

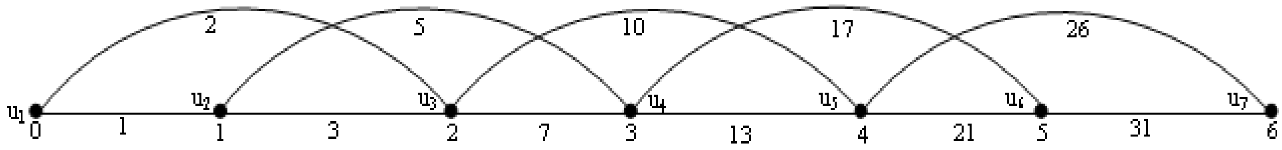

Figure-2: P_7^2

Theorem 2.5: P_n^3 is a mean square sum graph.

Proof: Let $u_1u_2\ldots u_n$ be the path P_n. Let G = P_n^3. Then V(G) = $\{u_1, u_2, \ldots, u_n\}$ and E(G) = $\{u_iu_{i+1}, u_ju_{j+2}, u_ku_{k+3} /1\leq i\leq n-1, 1\leq j\leq n-2, 1\leq k\leq n-3\}$. Clearly, G has n vertices and 3n-6 edges. Define f: V(G)$\rightarrow$\{0, 1, ..., n-1\} as follows $f(u_i) = i-1$, $1\leq i\leq n$. The induced function f*: E(G)$\rightarrow$N is defined by $f^*(u_iu_{i+1}) = i^2-i+1$, $1\leq i\leq n-1$; $f^*(u_iu_{i+2}) = i^2+1$, $1\leq i\leq n-2$; $f^*(u_iu_{i+3}) = i^2+i+2$, $1\leq i\leq n-3$ is injective. Hence P_n^3 is a mean square sum graph.

Example 2.6: A mean square sum labeling of P_5^3 is given in figure 3.

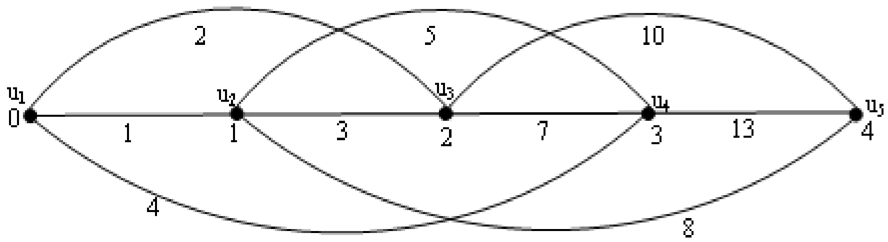

Figure-3: P_5^3

Theorem 2.7: The graph $D_2(P_n)$ is a mean square sum graph.

Proof: Let $u_1u_2\ldots u_n$ be a of path P_n and $v_1v_2\ldots v_n$ be another path P'_n. Join u_iv_{i+1}, v_iu_{i+1}; $1\leq i\leq n-1$. The resultant graph is $D_2(P_n)$. Let G = $D_2(P_n)$. Then V(G) = $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and E(G) = $\{u_iu_{i+1}, v_iv_{i+1}, u_iv_{i+1}, v_iu_{i+1} /1\leq i\leq n-1\}$. Clearly G has 2n vertices and 4(n-1) edges. Define f :V(G)$\rightarrow$\{0, 1, ..., 2n-1\} as follows $f(u_i) = 2i-2$, and $f(v_i) = 2i-1$, for $1\leq i\leq n$. The induced function f*: E(G)$\rightarrow$N is defined by $f^*(u_iu_{i+1}) = 4i^2-4i+2$, $f^*(v_iv_{i+1}) = 4i^2+1$, $f^*(u_iv_{i+1}) = 4i^2-2i+2$, $f^*(v_iu_{i+1}) = 4i^2-2i+1$, for $1\leq i\leq n-1$ is injective. Hence $D_2(P_n)$ is a mean square sum graph.

Example 2.8: A mean square sum labeling of $D_2(P_5)$ is given in figure 4.

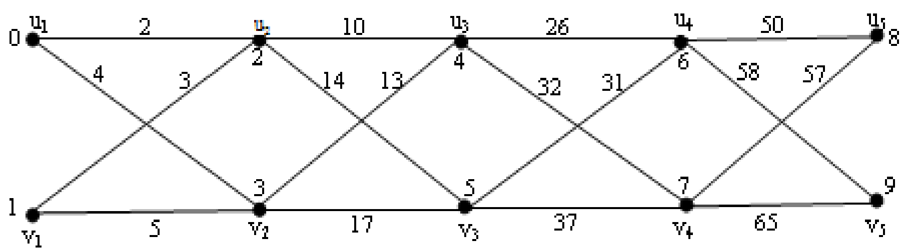

Figure-4: $D_2(P_5)$

Theorem 2.9: The splitting graph of path P_n is a mean square sum graph.

Proof: Let $u_1u_2\ldots u_n$ be the vertices of path P_n and $v_1v_2\ldots v_n$ be the newly added vertices to form the splitting graph of P_n. Let $G = S'(P_n)$. Then $V(G) = \{u_i, v_i /1\leq i\leq n\}$ and $E(G) = \{u_iu_{i+1}, u_iv_{i+1}, v_iu_{i+1} /1\leq i\leq n-1\}$. Also G has 2n vertices and 3n-3 edges. Define $f: V(G) \rightarrow \{0, 1, ..., 2n-1\}$ as follows $f(u_i) = 2i-2$, and $f(v_i) = 2i-1$, for $1\leq i\leq n$. The induced function $f^*: E(G)\rightarrow N$ is defined by $f^*(u_iu_{i+1}) = 4i^2-4i+2$, $f^*(u_iv_{i+1}) = 4i^2-2i+2$, $f^*(v_iu_{i+1}) = 4i^2-2i+1$, for $1\leq i\leq n-1$ is injective. Hence $S'(P_n)$ is a mean square sum graph.

Example 2.10: A mean square sum labeling of $S'(P_6)$ is given in figure 5.

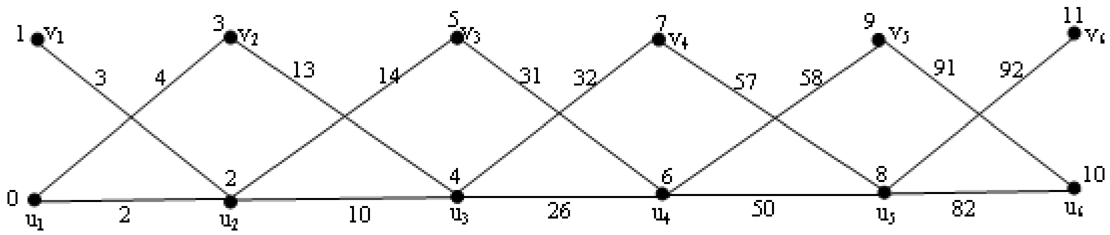

Figure-5: $S'(P_6)$

Theorem 2.11: Switching of a pendant vertex in path P_n is a mean square sum graph.

Proof: Let $u_1u_2\ldots u_n$ be the vertices of path P_n and G^v denotes the graph obtained by switching of a pendant vertex v of $G = P_n$. Without loss of generality let the switching vertex be $v = u_1$. Clearly $V(G^v) = \{u_1, u_2, \ldots, u_n\}$ and $E(G^v) = \{u_iu_{i+1}, u_1u_j \,/\, 2\leq i\leq n\text{-}1, 3\leq j\leq n\}$. Hence G has n vertices and 2n-4 edges. Define $f: V(G)\rightarrow\{0, 1, ..., p\text{-}1\}$ as follows $f(u_1) = 0$ and $f(u_i) = i\text{-}1$, $2\leq i\leq n$. The induced function $f^*: E(G)\rightarrow N$ is defined by $f^*(u_iu_{i+1}) = i^2\text{-}i\text{+}1$, $2\leq i\leq n\text{-}1$; $f^*(u_1u_i) = \left\lfloor\frac{(i-1)^2}{2}\right\rfloor$, $3\leq i\leq n$ is injective. Hence G^v is a mean square sum graph.

Example 2.12: A mean square sum labeling of G^v when n = 5 is given in figure 6.

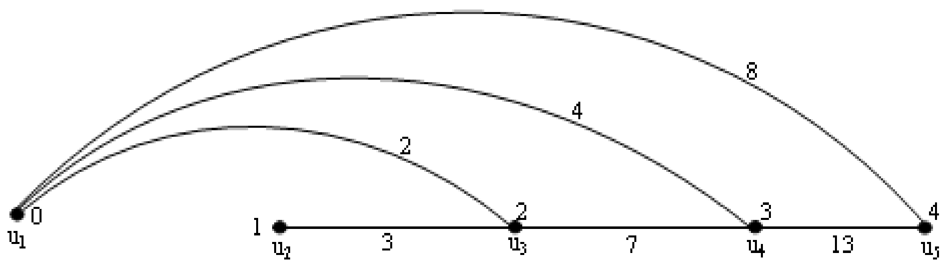

Figure-6

Theorem 2.13: Let P_n be the path and G be the graph obtained from P_n by attaching C_3 in both end edges of P_n. Then G is a mean square sum graph.

Proof: Let P_n the path $u_1u_2\ldots u_n$. Add two new vertices v_1 and v_2. Join v_1u_1, v_1u_2, v_2u_{n-1}, and v_2u_n. The resultant graph is G with $V(G)= \{u_1, u_2, \ldots, u_n, v_1, v_2\}$ and $E(G)= \{u_iu_{i+1}, u_1v_1, u_2v_1, u_{n-1}v_2, u_nv_2/1\leq i\leq n\text{-}1\}$. Then G has n+2 vertices and n+3 edges. Define $f: V(G)\rightarrow\{0, 1, ..., n\text{+}1\}$ as follows $f(u_i) = i$, $1\leq i\leq n$; $f(v_1)= 0$; $f(v_2) = n\text{+}1$. The induced function $f^*:E(G)\rightarrow N$ is defined by $f^*(u_iu_{i+1}) = i^2\text{+}i\text{+}1$, $1\leq i\leq n\text{-}1$; $f^*(u_1v_1) = 1$; $f^*(u_2v_1) = 2$; $f^*(u_{n-1}v_2) = n^2\text{+}1$; $f^*(u_nv_2) = n^2\text{+}n\text{+}1$ is injective. Hence G is a mean square sum graph.

Example 2.14: A mean square sum labeling of G when n = 8 is given in figure 7.

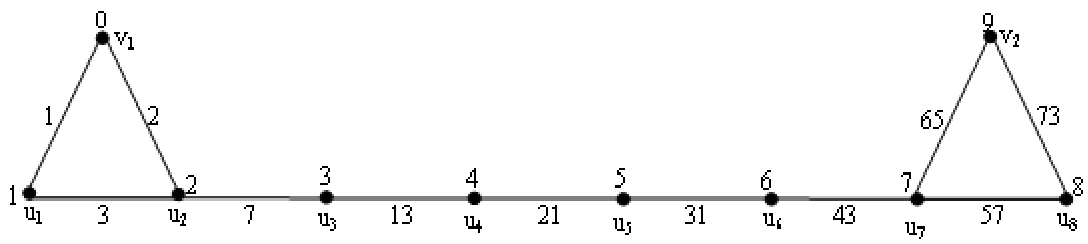

Figure-7

Theorem 2.15: A Triangular snake T_n is a mean square sum graph for $n \geq 3$.

Proof: Let $u_1, u_2, ..., u_n$ be the vertices of P_n. For $1\leq i\leq n\text{-}1$, add new vertex v_i which is joined to the vertices u_i and u_{i+1} of path P_n. The resultant graph $G = T_n$ with $V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, ..., v_{n-1}\}$ and $E(G) = \{u_iu_{i+1}, u_iv_i, u_{i+1}v_i /1\leq i \leq n\text{-}1\}$. Then G has 2n-1 vertices and 3n-3 edges. Define $f: V(G)\rightarrow\{0, 1, \ldots, 2n\text{-}2\}$ by $f(u_i) = 2i\text{-}2$, $1\leq i \leq n$; $f(v_i) = 2i\text{-}1$, $1\leq i\leq n\text{-}1$. The induced function $f^*: E(G)\rightarrow N$ is defined by $f^*(u_iv_i) = 4i^2\text{-}6i\text{+}3$, $f^*(u_{i+1}v_i) = 4i^2\text{-}2i\text{+}1$, and $f^*(u_iu_{i+1}) = 4i^2\text{-}4i\text{+}2$, for $1\leq i\leq n\text{-}1$ is injective. Hence a triangular snake T_n is a mean square sum graph.

Example 2.16: A mean square sum labeling of T_7 is given in figure 8.

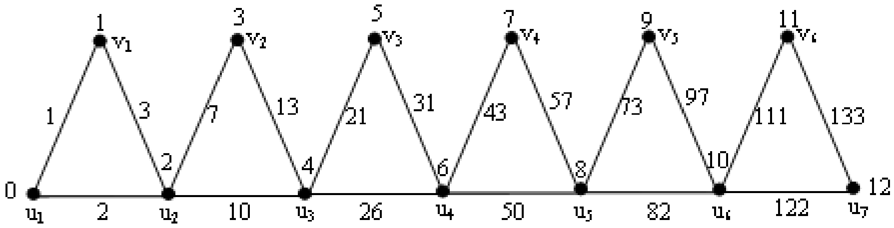

Figure-8: T_7

Theorem 2.17: A double triangular snake DT_n is a mean square sum graph.

Proof: Consider a path $u_1u_2...u_n$. Join u_i and u_{i+1} to two new vertices v_i and w_i, $1\leq i\leq n-1$. The resultant graph G = DT_n with V(G) = $\{u_i, v_i, w_i /1\leq i\leq n\}$ and E(G) = $\{u_iu_{i+1}, u_iv_i, u_{i+1}v_i, u_{i+1}w_i, u_iw_i /1\leq i \leq n-1\}$. Then G has 3n-2 vertices and 5n-5 edges. Define f :V(G)$\rightarrow\{0, 1, \ldots, 3n-3\}$ by $f(u_1) = 0$; $f(u_i) = 3i-5$, $2\leq i \leq n$; $f(v_i) = 3i-1$, $1\leq i\leq n-1$; $f(w_i) = 3i$, $1\leq i\leq n-1$. The induced function f*: E(G)$\rightarrow$N is defined by $f^*(u_1u_2) = 1$; $f^*(u_iu_{i+1}) = 9i^2-21i+14$, $2\leq i\leq n-1$; $f^*(u_1v_1) = 2$; $f^*(u_iv_i) = 9i^2-18i+13$, $2\leq i\leq n-1$; $f^*(u_{i+1}v_i) = 9i^2-9i+3$, $1\leq i\leq n-1$; $f^*(u_1w_1) = 4$; $f^*(u_iw_i) = 9i^2-15i+12$, $2\leq i\leq n-1$; $f^*(u_{i+1}w_i) = 9i^2-6i+2$, $1\leq i\leq n-1$ is injective. Hence the double triangular snake DT_n is a mean square sum graph.

Example 2.18: A mean square sum labeling of DT_6 is given in figure 9.

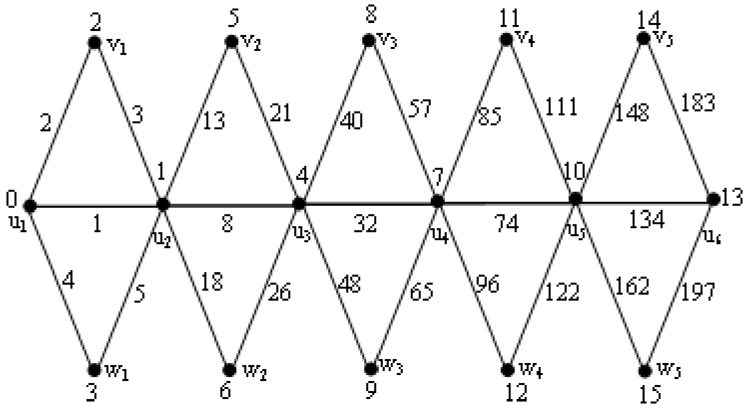

Figure-9: DT_6

Theorem 2.19: The coconut tree is a mean square sum graph.

Proof: Let $u_1u_2\ldots u_n$ be the vertices of path P_n and $v_1v_2\ldots v_m$ be the pendent vertices being adjacent with the vertex u_1. The resultant graph G is a coconut tree with V(G) = $\{u_1, u_2, \ldots, u_n, v_1, v_2, ..., v_m\}$ and E(G) = $\{u_iu_{i+1}, u_1v_j /1\leq i\leq n-1, 1\leq j\leq m\}$. Then G has n+m vertices and m+n-1 edges. Define f :V(G)$\rightarrow\{0, 1, ..., m+n-1\}$ as follows $f(u_i) = i-1$, $1\leq i\leq n$ and $f(v_i) = n-1+i$, $1\leq i\leq m$. The induced function *f*: E(G)→N* is defined by $f^*(u_iu_{i+1}) = i^2-i+1$, $1\leq i\leq n-1$; $f^*(u_1v_i) = \left\lfloor\frac{(n-1+i)^2}{2}\right\rfloor$, $1\leq i\leq m$ is injective. Hence the coconut tree is a mean square sum graph.

Example 2.20: A mean square sum labeling of G when n = 5, m = 6 is given in figure 10.

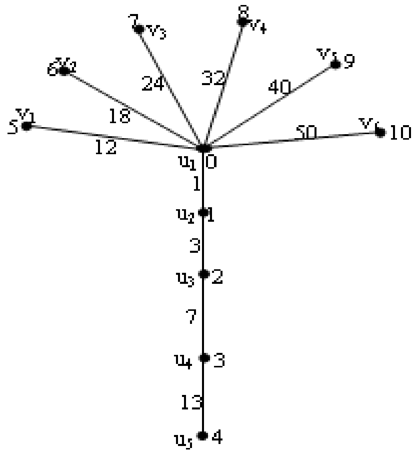

Figure-10

Theorem 2.21: Let G be a graph obtained by attaching each vertex of P_n to the central vertex of $K_{1,2}$. Then G is a mean square sum graph.

Proof: Let $u_1, u_2, \ldots, u_n$ be the path P_n and v_i, w_i, x_i be the vertices of $K_{1,2}$ such that the central vertex x_i is considered with u_i, $1 \le i \le n$. The resultant graph is G with $V(G) = \{u_i, v_i, w_i / 1 \le i \le n\}$ and $E(G) = \{u_iv_i, v_iw_i, u_ju_{j+1} / 1 \le i \le n, 1 \le j \le n-1\}$. Then G has 3n vertices and 3n-1 edges. Define $f: V(G) \to \{0, 1, \ldots, 3n-2\}$ by $f(u_i) = 3i-3$, $1 \le i \le n$; $f(v_i) = 3i-2$, $1 \le i \le n$; $f(w_i) = 3i-1$, $1 \le i \le n$. The induced function $f^*: E(G) \to N$ is defined by $f^*(u_iw_i) = 9i^2-12i+5$, $1 \le i \le n$; $f^*(u_iv_i) = 9i^2-15i+7$, $1 \le i \le n$; $f^*(u_iu_{i+1}) = 9i^2-9i+5$, $1 \le i \le n-1$ is injective. Hence G is a mean square sum graph.

Example 2.22: A mean square sum labeling of G when n = 4 is given in figure 11.

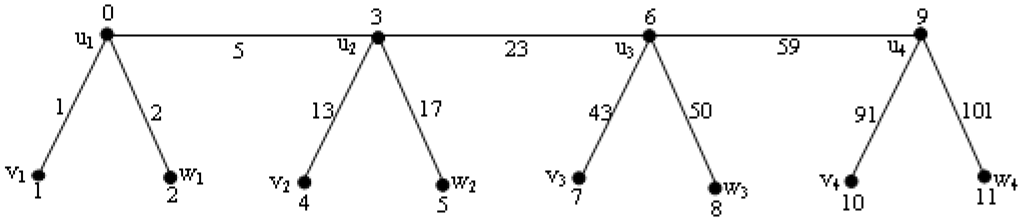

Figure-11

Theorem 2.23: Let G be a graph obtained by attaching each vertex of Pn to the central vertex of K1, 3. Then G is a mean square sum graph.

Proof: Let $u_1, u_2, \ldots, u_n$ be the path P_n and v_i, w_i, y_i, x_i be the vertices of $K_{1,3}$ such that the central vertex x_i is considered with u_i, $1 \le i \le n$. The resultant graph is G with *V(G)* $= \{u_i, v_i, w_i, y_i / 1 \le i \le n\}$ and *E(G)*$= \{u_iv_i, v_iw_i, u_iy_i, u_ju_{j+1} / 1 \le i \le n, 1 \le j \le n-1\}$. Then G has 4n vertices and 4n-1 edges. Define $f: V(G) \to \{0, 1, \ldots, 4n-1\}$ by $f(u_i) = 4i-4$, $1 \le i \le n$; $f(v_i) = 4i-3$, $1 \le I \le n$; $f(w_i) = 4i-2$, $1 \le I \le n$; $f(y_i) = 4i-1$, $1 \le I \le n$. The induced function $f^*: E(G) \to N$ is defined by $f^*(u_iv_i) = 16i^2-28i+13$, $1 \le I \le n$; $f^*(u_iw_i) = 6i^2-24i+10$, $1 \le I \le n$; $f^*(u_iy_i) = 16i^2-20i+9$, $1 \le I \le n$; $f^*(u_iu_{i+1}) = 16i^2-16i+8$, $1 \le I \le n-1$ is injective. Hence G is a mean square sum graph.

Example 2.24: A mean square sum labeling of G when n = 4 is given in figure 12.

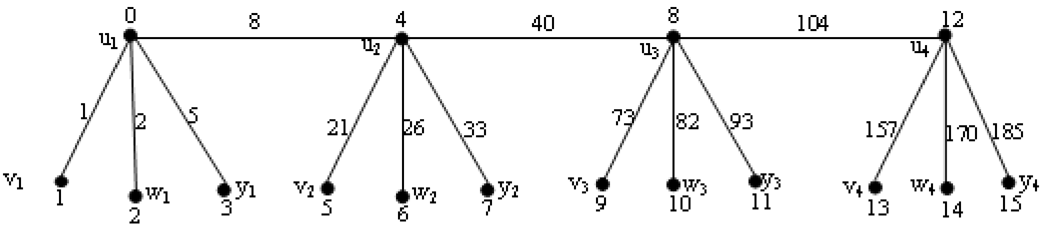

Figure-12

REFERENCES

1. F. Harary, 1988, Graph theory, Narosa Publishing House, New Delhi.
2. J. A. Gallian, A Dynamic Survey of Graph Labeling, The Electronics Journal of Combinatorics, Vol. 17, (#DS6), 2014. URL: http://www.combinatorics.org/Surveys/ds6.pdf
3. C. Jayasekaran, S. Robinson Chellathurai and M. Jaslin Melbha, Mean Square Sum labeling of some graphs, Mathematical Sciences International Research Journal, Vol. 4, (2), 2015, 238-242.
4. C. Jayasekaran, S. Robinson Chellathurai and M. Jaslin Melbha, Mean Square Sum labeling of some cycle related graphs, Mathematical Sciences International Research Journal, Vol. 4, (2), 2015, 257-262.
5. S. Somasundaram and R. Ponraj, Mean labeling of graphs, National Academy of Science Letters, Vol.26, (1), 2003, 210-213.
6. V. Ajitha, S. Arumugam and K.A. Germina, On Square sum graphs, AKCE J. Graphs Combin., Vol.6, (1), 2009, 1- 10.

Source of support: Nil, Conflict of interest: None Declared