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SOME RESULTS ON COMPLETELY EXTENDABLE AND WEIGHTED EXTENDABLE GRAPHS

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ABSTRACT

In this paper we study certain properties of completely extendable graphs. In [3] we define an extension in a graphG by adding edges in a particular pattern. Certain graphs become complete after the finite extensions. Such graphs are called completely extendable graphs. Necessary and sufficient condition for a graph to be completely extendable is given in [3]. Here we discuss theorems related to completely extendable graphs. We also consider weighted graph and define extensions on weighted graph. We also give characterisation for a weighted graph $G$to be completely weighted extendable. Next we consider G which is not completely extendable. We define deficiency number of a graph which is not completely extendable and give a particular pattern of extension to make such graphs complete.

Keywords: *Weighted graphs, Completely weighted extendable graphs, Deficiency number.*

Subject Classification: *05C.*

1. INTRODUCTION

Consider a (p,q) graph which is completely extendable. Necessary and sufficient condition for a graph G to be completely extendable is given [3]. In this paper we discuss some theorems related to edge density and average degree of a completely extendable graph. We also define a weighted graph and extensions in weighted graphs. Extensions in a weighted graph is defined as adding one edge of weight 1 in the first extension, adding two edges of weight 2 in the second extension and so on. Some weighted graphs become complete after a finite number of extensions. Further, we give necessary and sufficient condition for a weighted graph to become a completely weighted extendable graph. Then we consider a $(p,\ q)$ graph which is not completely extendable and define deficiency number for such graphs. Finally, we introduce a typical pattern of adding edges in each extension based on the deficiency number to make the graph complete. For basic definitions and results in Graph Theory, we follow [2].

2. PRELIMINARIES

Definition 2.1[3]: Let G be a simple (p,q) graph. Extension on G is defined as follows; in the first extension, add one edge to G, denoted as G^1, $G^1 = G \cup \{e_1\}$. In the second extension add two edges on G^1 denoted by G^2, $G^2 = G \cup \{e_1, e_2, e_3\}$ and so on until no such an extension remains.

Definition 2.2[3]: If $G^k \cong K_p$, then G is said to be a completely extendable graph and n is known as the order of extension.

Theorem 2.3 [3]: Let G be a (p,q) graph. If $q = pk - r$ where $r = k\,(k+1)/2$ and $k < p$, then G is completely extendable. Order of extension is $p-(k+1)$.

Theorem 2.4 [3]: Let G be a (p,q) graph and let $G^k = G \cup \{e_1, e_2 \ldots . e_m\}$. If G^k is the k^{th} extension of G, then $m = \frac{k(k+1)}{2}$.

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Definition 2.5[1]: Average degree of a graph$G$, $\bar{d}(G)$ is defined as sum of the degree of vertices in G divided by total number of vertices, $\bar{d}(G) = \sum_{v \epsilon V} \frac{d(v)}{|V(G)|}$.

Definition 2.6: (Edge Density) [1] Let G be a(p, q) graph. Edge density of G is defined as number of existing edges divided by number of possible edges. Edge density is denoted by $\mu(G)$, $\mu(G) = \dfrac{2q}{p(p-1)}$.

Remark 2.7: $\mu(G)$ lies between 0 and 1. If G is an empty graph, then$\mu(G)$ is 0 and if G is a complete graph, then $\mu(G)$ is 1.

3. MAIN RESULTS

Theorem 3: 1 Let G be a (p, q) graph, $\bar{d}(G)$ is the average degree of G. G is completely extendable if and only if $\bar{d}(G) + \dfrac{n(n+1)}{p} = p - 1$, where n is the order of extension and $n \leq p - 2$.

Proof: First assume that G is completely extendable and n be the order of extension. Then $G^n \cong K_p$. Average degree of $K_p = \frac{p\,(p-1)}{p} = p - 1$. Since we add one edge in G^1, average degree of $G^1 = \bar{d}(G) + 2/p$. Since we add 2 edges in G^1 to get G^2, average degree of $G^2 = \bar{d}(G) + 2/p + 4/p$. Like this average degree of G^n,

$$\begin{aligned}\bar{d}(G^n) &= \bar{d}(G) + 2/p + 4/p + 6/p + \ldots\ldots\ldots 2n/p = p-1.\\ &= \bar{d}(G) + 2/p[1+2+\ldots.n] = p-1\\ &= \bar{d}(G) + \frac{2}{p}\frac{n(n+1)}{2} = p-1\\ &= \bar{d}(G) + \frac{n(n+1)}{p} = p-1.\end{aligned}$$

Conversely assume that

$$\begin{aligned}&\bar{d}(\mathrm{G}) + \frac{n(n+1)}{p} = p-1.\\ &\sum \frac{d(v)}{p} + \frac{n(n+1)}{p} = p-1\\ &\sum \frac{d(v)}{p} = (p-1) - \frac{n(n+1)}{p}\\ &\sum \frac{d(v)}{p} = \frac{p(p-1) - n(n+1)}{p}\\ &\sum d(v) = p(p-1) - n(n+1).\end{aligned}$$

By first theorem of graph theory$2q = p(p-1) - n(n+1)$. That is $q = \dfrac{p(p-1) - n(n+1)}{2}$ (1)

For any value of p and n in (1),q can be expressed in the form $kp - r$. Then by theorem 2.3 G is a completely extendable graph.

Example: For $p = 50$, $n = 15$ from (1) $q = 1105 = 50 \times 34 - \dfrac{34 \times 35}{2}$.

Order of extension is $50 - (34 + 1) = 15$

Theorem 3.2: Let G be a (p, q) graph. $G$is completely extendable if and only if

$\mu(G) = 1 - \dfrac{n(n+1)}{p(p-1)}$, where n is the order of extension and $n \leq p - 2$.

Proof: Assume that G is completely extendable and n is the order of extension.

Then $q + \dfrac{n(n+1)}{2} = \dfrac{p(p-1)}{2}$ (a)

Dividing (a) by $\dfrac{p(p-1)}{2}$, we get $\dfrac{2q}{p(p-1)} + \dfrac{n(n+1)}{p(p-1)} = 1$

$\dfrac{2q}{p(p-1)} = 1 - \dfrac{n(n+1)}{p(p-1)}$. That is, $\mu\,(G) = 1 - \dfrac{n(n+1)}{p(p-1)}$.

Conversely assume that $\mu(G) = 1 - \dfrac{n(n+1)}{p(p-1)}$,

which implies $\dfrac{2q}{p(p-1)} + \dfrac{n(n+1)}{p(p-1)} = 1$

$2q + n(n+1) = p - 1$ That is $q + \dfrac{n(n+1)}{2} = \dfrac{p(p-1)}{2}$, which implies G is completely extendable and order of extension is n.

Theorem 3.3: Every empty graph is completely extendable.

Proof: Let G be a $(p, 0)$ graph. Number of edges added to G to get K_p is $\dfrac{p(p-1)}{2}$. That is

$$G \cup \{e_1, e_2, \ldots\ldots e_{\frac{p(p-1)}{2}}\} \cong K_p.$$

Then by theorem 2.4, $G^{p-1} = G \cup \{e_1, e_2, \ldots, e_{\frac{p(p-1)}{2}}\}$. That is, $G^{p-1} \cong K_p$ and $p - 1$ is the order of extension.

Definition 3.4 (*Weighted graph*): Let G be a graph with order p and size q. Let f be a weight function $f : V\,(G) \rightarrow \{1, 2\ldots p\}$ such that $f\,(v_i) = i$. That is, assigning weights for each vertex from 1 to p in a clockwise or anticlockwise direction. Define a weight function $f^*: E(G) \rightarrow \{1, 2, \ldots p-1\}$ such that $f^*(u, v) = |\,f(u) - f(v)\,|$, where $(u, v) \in E(G)$. If $e = u$ v then weight of e, $w\,(e) = f^*(u, v)$.

Example: Consider the following graph,

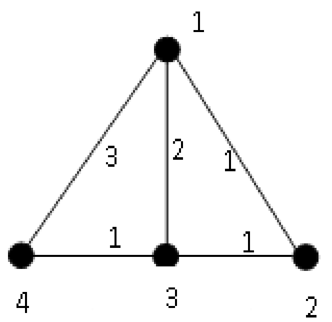

Figure-1

Definition 3.5 (***Weight of a graph***)**:** Let G be a weighted graph. Let $w(e_1), w(e_2), \ldots, w(e_q)$ be the weights of the edges $e_1, e_2, \ldots, e_q$. Weight of G is defined as the sum of the weight of edges of G. Weight of G, denoted by

$$w\,(G) = \sum_{i=1}^{q} w(e_i).$$

From figure1, Weight of G, $w\,(G) = 1 + 1 + 1 + 2 + 3 = 8$

Theorem 3.6: Weight of a complete graph $K_p$is $\sum_{i=1}^{p-1}(p-i)i$.

Proof: Total number of edges in K_p is $\frac{p(p-1)}{2}$. The edges are$(v_1,v_2),(v_1,v_3),\ldots,(v_1,v_p)$, $(v_2,v_3),\ldots,(v_2,v_p),\ldots,(v_{p-1},v_p)$. In K_p, Number of edges with weight 1 is $p-1$,number of edges with weight 2 is $p-2$,number of edges with weight 3 is $p-3$... number of edges with weight $p-2$ is 2, and number of edges with weight $p-1$ is 1.Then the total weight of a complete Graph$K_p = 1\times(p-1) + 2\times(p-2) + 3\times(p-3) + \ldots + (p-2)\times 2 + (p-1)\times 1 = \sum_{i=1}^{p-1}(p-i)i$.

Example: ConsiderK_6.$(1,2,3,4,5,6)$are the weights of the vertices .Total number of edges in K_6 is 15.Number of edges with weight 1 is 5, weight 2 is 4, weight 3 is 3, weight 4 is 2 and weight 5 is 1.

Total weight = $\sum_{i=1}^{5}(p-i)i$= 5×1+4×2 +3×3 +2×4 +1×5 = 35

Definition 3.7 (*Extension of a weighted Graph*): Let G be a weighted graph with p vertices and $q$edges.Extension of a weighted graph is defined as in the first extension G^1, add one edge of weight 1, in the second extension G^2, add 2 edges of weight 2 and so on.

Definition 3.8 (*Completely weighted extendable graph*): G is said to be completely weighted extendable graph if it is possible to add n edges of weight n in the n^{th} extension and $G^n \cong K_p$.n is called the order of the extension.

Example:

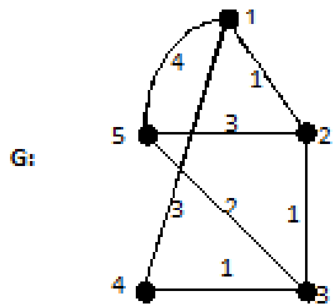

Figure-2: weight of the edge (5, 3) is 2.

Consider the weighted graph depicted in figure (2). G is a weighted graph with 5 vertices and 7 edges. That is, $2p-3$ edges. By theorem (2.3), G is completely extendable and order of extension is 2. In the first extension G^1, add the edge (4,5). In the second extension G^2, add 2 edges of weight 2. That is add (2, 4) and (1, 3). Then G becomes a complete graph. Therefore G is a completely weighted extendable graph and order of extension is 2.

Theorem 3.9: Tree with $p \geq 5$ is not completely weighted extendable.

Proof: By theorem (2.3), every tree is completely extendable and order of extension is $p-2$. If it is possible to add $p-2$ edges of weight $p-2$ in G^{p-2} and $G^{p-2} \cong K_p$, then the tree is completely weighted extendable. In a complete graph, number of edges with weight $p-2$ is at most 2. For $p \geq 5$, $p-2$ is greater than 2. Therefore extension is not possible up to a complete graph for a tree with $p \geq 5$.

Theorem 3.10: Let G be a (p,q) graph. If G is completely weighted extendable then $w(G) = w(K_p) - \sum_{i=1}^{n} i^2$, where n is the order of extension and size of G is $kp-r$ where $k < p \leq 2k+2$.

Proof: Given G be a completely weighted extendable graph with order of extension as n, that is,$G^n \cong K_p$, then it is possible to add one edge of weight 1 in the first extension G^1, in the second extension add 2 edges of weight 2, etc in the n^{th}extension add n edges of weight n. G became complete after the n^{th}extension.Total weight of the edges added to G

$$= 1 + (2+2) + (3+3+3) + \ldots\ldots (n+n\ldots\ldots+n) = 1\times 1 + 2\times 2 + 3\times 3 + \ldots\ldots n\times n = \sum_{i=1}^{n} i^2$$

Since n is the order of extension of G, $G + \{e_1, e_{.2}\ldots\ldots e_{\frac{n(n+1)}{2}}\} \cong K_p$

That is $w\,(G) + \sum_{i=1}^{n} i^2 \;=\; w\,(\,K_p\,)$ which implies $w\,(G) = w\,(K_p) - \sum_{i=1}^{n} i^2$.

Given that size of G is $kp - r$. By theorem 2.3, G is completely extendable. We have to prove that for $p > 2k + 2G$ is not completely weighted extendable. In a complete graph number of edges of weight 1 is $p - 1$, number of edges of weight 2 is $p - 2$ etc number of edges of weight $p - 2$ is 2 and edges with weight $p - 1$ is 1. That is number of edges of weight $p - (k + 1)$ is at most $k + 1$. G becomes completely weighted extendable if $p - (\,k + 1) \leq (k + 1)$. That is $p \leq 2k + 2$.

Example: Consider G with size $kp - r$ where $k = 1$ and $p = 5$ and order of extension of G is $p - 2 = 3$. G becomes a completely weighted extendable graph only if it is possible to add 3 edges of weight 3 in G^3. But in $K_{5,}$ number of edges of weight 3 is at most 2. Therefore extension is not possible in G when $p > 2k + 2$.

Theorem 3.11: Let G be a weighted graph. If $w\,(G) = w\,(K_p) - \sum_{i=1}^{p-(k+1)} i^2$ and size of G is $kp - r$, where $k < p \leq 2k + 2$,then G is completely weighted extendable.

Proof: Let Gbe a weighted graph having size $kp - r$. By theorem (2.3), G is completely extendable and order of extension is $p - (k + 1)$ (say n). Since G is a weighted graph add edges in such a way that in the first extension add one edge of weight 1, in the second extension add two edges of weight 2 ... n edges of weight n in the n^{th}extension and $G^n \cong K_p$. (n is the order of extension). Total weight of edges added in each extension is $\sum_{i=1}^{n} i^2$. Given that $w\,(G) = w\,(\,K_p) - \sum_{i=1}^{n} i^2$. That is $w\,(G) + \sum_{i=1}^{n} i^2 = w\,(K_p)$.

That is extension is possible and G becomes a complete graph in G^n. Therefore G is a completely weighted extendable graph.

Remark: If $p > 2k + 2$ extension is not possible. For example, consider a weighted graph G with size $kp - r$ where $p = 7$ and $k = 2$ then order of extension is $p - 3 = 4$. But G with 7 vertices have at most 3 edges of weight 4. Therefore extension is not possible.

Next, we give characterization of a weighted graph.

Theorem 3.12: Let G be a weighted graph .G is completely weighted extendable if and only if
$w\,(G) = w(\,K_p) - \sum_{i=1}^{p-(k+1)} i^2$ and size of G is $pk - r$ where $k < p \leq 2k + 2$.

Proof: Proof follow from theorems (3.10) and (3.11)..

Theorem 3.13: Let G be a weighted graph with p vertices and $kp - r$ edges. Let $G_1, G_2, \ldots, G_{p-1}$ are the edge disjoint subgraphs of G with edges of weights $1, 2, 3\ldots\ldots p - 1$ respectively where $G = G_1 \cup G_2 \cup\ldots\cup G_{p-1}$. Then G is completely weighted extendable only if number of edges in G_1 is exactly $p - 2$, in G_2 is exactly $p - 4$ etc, in G_n is exactly $p - 2n$, in $G_{\,n+1}$ is $p - (n + 1)$ ... $G_{p-2}$is 2 and G_{p-1} is 1, where $n = p - (k + 1)$.

Proof: By theorem 2.3, G with size $kp - r$ is completely extendable and order of extension is $p - (\,k + 1)$ (say n). Weighted graph G is said to be completely weighted extendable only if it is possible to add n edges of weight n in G^n. Since total number of edges of weight 1 in a complete graph is $p - 1$, first extension G^1 is possible only if G has $p - 2$ edges of weight 1. Like this $G^2$is possible only if G has exactly$p - 4$ edges of weight 2, etc G^n is possible only if G has exactly $p - 2n$ edges of weight n, $p - (n + 1)$ edges of weight $(n + 1)$, 2 edges of weight $p - 2$ and 1 edge of weight $(\,p - 1)$. Then the edge disjoint subgraphs $G_1, G_2 \ldots G_{p-1}$ with weights $1, 2, \ldots p - 1$ contains $(p - 2), (p - 4), \ldots$
$(p - 2n), (p - (n + 1)), \ldots$ 2,1 edges respectively.

Definition 3.14: (**Deficiency number)** Let G be a (p, q) graph which is not completely extendable. Let r be the maximum possible number of extension in G such that G^r is not complete. Then the deficiency number of G is defined as the number of edges required to make G^r complete.

Example: Consider the following graph G

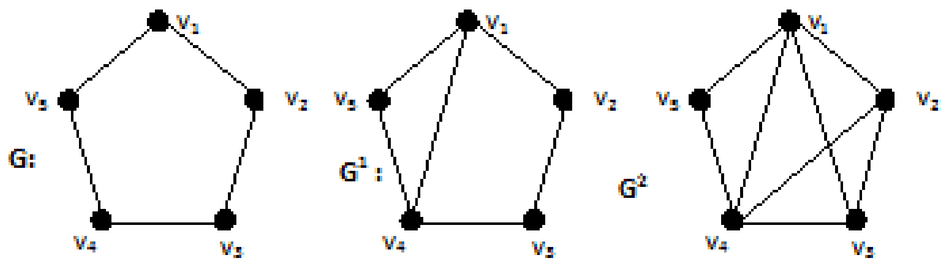

Figure-3

G^1 and G^2 are the possible extensions of G but $G^2$is not complete. Number of edges added to G^2 to get a complete graph is 2. Therefore 2 is the deficiency number of G.

Theorem 3.15: Let G be a (p, q) graph which is not completely extendable. If $1 \leq q \leq p-2$, then $p - (q+1)$ is the deficiency number of G in $(p-2)^{th}$ extension.

Proof: By theorem 3.3, If $q = 0$, then G is completely extendable and order of extension is$(p - 1)$. If $q = 1$, $(p - 2)$ is the maximum possible number of extension of G. But G^{p-2} is not complete. Number of edges added to G^{p-2} to get a complete graph is $(p - 2)$. Therefore, when $q = 1$, the deficiency number of G is $(p - 2)$. When $q = 2, (p - 2)$ is the maximum possible number of extension of G but G^{p-2} is not complete. $(p - 3)$ edges is required to make G^{p-2} a complete graph. Therefore for $q = 2$, $p - 3$is the deficiency number of G in the $(p-2)^{th}$extension. If $q = p - 2$, then 1 edge is required to make G^{p-2} a complete graph. Therefore 1 is the deficiency number in the $(p-2)^{th}$ extension. In general for$1 \leq q \leq p-2$,$p - (q+1)$ is the deficiency number of$G$ in the $(p - 2)^{th}$ extension.

Example: Consider a graph with 6 vertices and 4 edges. Then number of edges added to G to get a complete graph is 11. In G^1 add one edge, in G^2add 2 edges, in G^3 add 3 edges, in G^4 add 4 edges. Total number of edges added is $1 + 2 + 3 + 4 = 10$, which implies only one edge is required to get make G complete. Therefore, 5th extension is not possible. That is, 4 is the maximum possible number of extension in G and 1 is the deficiency number of G in the 4th extension.

Theorem 3.15: Let$G$ be a (p, q) graph which is not completely extendable. If the size of G is q where $1 \leq q \leq p-2$ then G can be extended completely by adding edges in such a way that for G^1 add 2 edges in G, for G^2add 3 edges in G^1 etc, for G^madd $m+1$ edges in G^{m-1} and for G^{m+1}add $m+1$ edges, for G^{m+2}add$m+2$ edges etc, for G^{p-2} add p-2 edges and $G^{p-2} \cong K_p$ where $m = p - (q+1)$.

Proof: Given that G be a (p, q) graph which is not completely extendable. If $1 \leq q \leq p-2$ then by theorem 3.14, $p-(q+1)$ is the deficiency number of G in the $(p - 2)^{th}$ extension. We can check this theorem by example. For this, consider a graph G with 7 vertices and 3 edges. Then the number of edges added to G to get a complete graphK_7 is 18. Deficiency number of G is $7 - (3+1) = 3$. In G^1 add 2 edges, in G^2 add 3 edges, in G^3 add 4 edges, in G^4 add 4 edges, in G^5 add 5 edges. Total number of edges added is $2+3+4+4+5 = 18$. That is $G^5 \cong K_7$.

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