International Journal of Mathematical Archive-7(9), 2016, 77-82
Available online through www.ijma.info ISSN 2229 – 5046

$d - \beta - \psi -$ CONTRACTIVE TYPE MAPPINGS IN DISLOCATED METRIC SPACES

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(Received On: 06-09-16; Revised & Accepted On: 22-09-16)

ABSTRACT

In this paper, we introduce d-β-ψ-contractive mappings in the context of dislocated metric spaces. Further, we give some fixed point results using this property and also discuss these results for the setting of cyclic contractions. In the end we discuss some examples to illustrate our results.

1. INTRODUCTION

Fixed point theory is one of the most dynamic research subject in non-linear analysis. The roots of fixed point theory lies in work of Brouwer and Banach. In 1912 Brouwer [1] proved a result that a unit closed ball in $\mathbb{R}^n$ has a fixed point. The most remarkable result in fixed point theory was given by Banach [2] in 1922. He proved that each contraction in a complete metric space has a unique fixed point. Later on, many authors generalized the Banach fixed point theorem in various ways [3-8]. Recently, Samet *et al.* [9] introduced the notion of α-ψ contractive mappings and proved the related fixed point theorems.

The generalization of well known Banach Contraction Principle of metric space to the dislocated metric space proved by P. Hitzler and A. K. Seda, played a key role in the development of logic programming semantics [10]. This concept of dislocated metric space was further generalized into dislocated quasi, right dislocated, left dislocated metric spaces by M.A. Ahmed *et.al* ([11], [12]).

In this paper we generalize the concept of α-ψ-contractive mappings in the setting of dislocated metric space, as d-β-ψ-contractive mappings.

2. PRELIMINARIES

Definition 2.1 Let Ψ be a family of functions $\psi\colon [0,\infty) \to [0,\infty)$ satisfying the following conditions:
(i) ψ is nondecreasing;
(ii) There exist $k_0 \in \mathbb{N}$ and $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that
$\psi^{k+1}(t) \le a\psi^k(t) + v_k$

For $k \ge k_0$ and any $t \in \mathbb{R}^+$

Lemma 2.2: ([13]) $If\ \psi\ \in\ \Psi$, then the following hold:
(i) $\left(\psi^n(t)\right)_{n\epsilon\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in \mathbb{R}^+$
(ii) $\psi(t) < t$ for any $t \in (0,\infty)$
(iii) ψ is continuous at 0;
(iv) the series $\sum_{n=1}^{\infty} \psi^k(t)$ converges for any $t \in \mathbb{R}^+$
Recently, Samet *et al.* [9] introduced the following concepts.

Definition 2.3: let $T\colon X \to X$ and $\alpha : X \times X \to [0,\infty)$. we say that T is α-admissible if for all$x, y\ \epsilon\ X$, we have
$$\alpha(x,y) \ge 1 \quad \Longrightarrow \quad \alpha(Tx,Ty) \ge 1$$

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Definition 2.4: let (X, d) be a metric space and let $T : X \to X$ be a given mapping. We say that T is an $\alpha-\psi$-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$$

For all $x, y \epsilon X$

Clearly, any contractive mapping, that is, a mapping satisfying Banach contraction, is an $\alpha-\psi$-contractive mapping with $\alpha(x, y) = 1$ for all $x, y \epsilon X$ and $\psi(t) = kt$, for all $t > 0$ and some $k \in [0,1)$.

Various examples of such mappings are presented in [9]. The main results in [9] are the following fixed point theorems:

Theorem 2.5: Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping. Suppose that
(i) T is α-admissible;
(ii) There exist $x_0 \epsilon X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then there exist $u \epsilon X$ such that $Tu = u$.

Theorem 2.6: Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping. Suppose that
(i) T is α-admissible;
(ii) There exist $x_0 \epsilon X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.
Then there exist $u \in X$ such that $Tu = u$.

Theorem 2.7: Adding the condition (iv) to the hypothesis of the Theorem 1.5 and Theorem 1.6 we obtain the uniqueness of a fixed point of T.
(iv) For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$
Hitzler and Seda [10] introduced the concept of dislocated metric space (d-metric space) as follows:

Definition 2.8: Let X be a non empty set and let $d: X \times X \to [0, \infty)$ be a function satisfying the following conditions:
(1) $d(x, y) = d(y, x)$
(2) $d(x, y) = d(y, x) = 0 \; implies \; x = y$
(3) $d(x, y) \leq d(x, z) + d(z, y) for \; all \;\; x, y, z \in X$
Then d is called dislocated metric (or simply d-metric) on X and the pair (X, d) is called dislocated metric space.

Example 2.9: Let (X, d) be a metric space. The function $f: X \times X \to \mathbb{R}^+$, defined as
$d(x, y) = \max(x, y)$; for all $x, y \in X$ is a d-metric on X.

Definition 2.10: A sequence $\{x_n\}$ in a d-metric space (X, d) is said to be d-convergent if for every given $\in > 0$ there exist an $n \in N$ and $x \in X$ such that $d(x_n, x) < \in$ for all $n > N$ and it is denoted by $\lim_{n\to\infty} x_n = x \; or \; x_n \to x \; as \; n \to \infty$

Definition 2.11: A sequence $\{x_n\}$ in a d-metric space (X, d) is said to be d-Cauchy sequence if for every $\in > 0$ there exist $n_0 \in N$ such that $d(x_n, x_m) < \in$ for all $m, n \in N$

Definition 2.12: A d-metric space (X, d) is called complete if every Cauchy sequence is convergent.

Lemma 2.13: let (X, d) be a d-metric space, (x_n) be a sequence in X and $x \in X$. Then $x_n \to x(n \to \infty)$ if and only if $d(x_n, x) \to 0(n \to \infty)$

Lemma 2.14: let (X, d) be a d-metric space and let (x_n) be a sequence in X. If the sequence (x_n) is convergent then the limit point is unique.

Theorem 2.15: Let (X, d) be a complete d-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$:

$$d(Tx, Ty) \leq kd(x, y), \tag{1}$$

where $k \in [0,1)$. then T has a unique fixed point.

3. MAIN RESULTS

We introduced the concept of d-β-ψ-contractive mapping as follows:

Definition 3.1: Let (X,d) be a d-metric space and let $T:X\to X$ be a given mapping. We say that T is a d-β-ψ-contractive mapping if there exist two functions $\beta: X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that for all $x,y\in X$, we have

$$\beta(x,y)d(Tx,Ty)\le\psi(d(x,y)). \tag{2}$$

Definition 3.2: Let $T:X\to X$ and $\beta: X\times X\to[0,\infty)$. We say that T is a β-admissible if for all $x,y\in X$, we have

$$\beta(x,y)\ge 1 \quad\Longrightarrow\quad \beta(Tx,Ty)\ge 1$$

Theorem 3.3: Let (X,d) be a d-metric space and let $T:X\to X$ is a d-β-ψ-contractive mapping and satisfies the following conditions:

(i) T is a β-admissible
(ii) there exist $x_0 \in X$ such that $\beta(x_0,Tx_0)\ge 1$;
(iii) T is d-continuous.

Then there exists $u\in X$ such that $Tu=u$.

Proof: Let $x_0\in X$ such that $\beta(x_0,Tx_0)\ge 1$ (such points exist from the condition (ii)). Define the sequence $\{x_n\}$ in X by $x_{n+1}=Tx_n$ for all $n\ge 0$. If $x_{n0}=x_{no+1}$ for some n_0, then $u=x_{n0}$ is a fixed point of T. So, we can assume that $x_n\ne x_{n+1}$ for all n. Since T is β-admissible, we have

$$\beta(x_0,x_1)=\beta(x_0,Tx_0)\ge 1 \quad\Longrightarrow\quad \beta(Tx_0,Tx_1)=\beta(x_1,x_2)\ge 1.$$

Inductively, we have

$$\beta(x_n,x_{n+1})\ge 1 \text{ for all } n=0,1,2\ldots\ldots. \tag{3}$$

From (2) and (6), it follows that for all $n\ge 1$, we have

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n)\\ &\le \beta(x_{n-1},Tx_{n-1})d(Tx_{n-1},Tx_n)\\ &\le \psi(d(x_{n-1},x_n))\end{aligned}$$

$$d(x_n,x_{n+1})\le\psi^n(d(x_0,x_1)) \quad \text{for all } n\ge 1 \tag{4}$$

Using (5), we have

$$\begin{aligned} d(x_n,x_m) &\le d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})+\cdots+d(x_{m-1},x_m)\\ &\le \sum_{k=n}^{m-1} d(x_k,x_{k+1})\\ &\le \sum_{k=n}^{m-1}\psi^k(d(x_0,x_1))\end{aligned}$$

Since $\psi\in\Psi$ and $d(x_0,x_1)>0$, by lemma 1.2, we get $\sum_{k=1}^{\infty}\psi^k d(x_0,x_1)<\infty$

Thus, we have $\lim_{n,m\to 0} d(x_n,x_m)=0$.

This implies that $\{x_n\}$ is a Cauchy sequence in the d-metric space (X,d). since (X,d) is complete, there exist $u\in X$ such that $\{x_n\}$ is d-convergent to u. since T is d-continuous, it follows that $\{Tx_n\}$ is d-convergent to Tu. by the uniqueness of the limit, we get $u=Tu$, that is, u is a fixed point of T.

The next theorem does not require continuity.

Theorem 3.5: Let (X,d) be a complete d-metric space. Suppose that $T:X\to X$ is a d-β-ψ-contractive mapping and the following conditions satisfies :

(i) T is β-admissible;
(ii) there exists $x_0\in X$ such that $\beta(x_0,Tx_0)\ge 1$;
(iii) If $\{x_n\}$ is a sequence in X such that $\beta(x_n,x_{n+1})\ge 1$ for all n and $\{x_n\}$ is a d-convergent to $x\in X$, then $\beta(x_n,x)\ge 1$ for all n.

Then there exist $u\in U$ such that $Tu=u$.

Proof: Following the proof of theorem 2.4, we know that the sequence $\{x_n\}$ defined by $x_{n-1}=Tx_n$ for all $n\ge 0$ is a Cauchy sequence in the complete metric space (X,d) that is d-convergent to $u\in X$, From (6) and (iii), we have

$$\beta(x_n,u)\ge 1 \quad \text{for all } n\ge 0. \tag{5}$$

Using the basic properties of d-metric together with (2) and (5), we have

$$\begin{aligned} d(x_{n+1}, Tu) &\leq d(Tx_n, Tu) \\ &\leq \beta(x_n, u) d(Tx_n, Tu) \\ &\leq \psi d(x_n, u) \end{aligned}$$

Letting $n \to \infty$, and since ψ is continuous at $t = 0$, it follows that

$$d(u, Tu) = 0$$

By definition, we obtain $u = Tu$

With the help of following example, we show that the hypotheses in theorems 2.4 and 2.5 do not guarantee uniqueness of fixed point.

Example 3.6: Let $X = [0, \infty)$ be the d-metric space, where $d(x, y) = \max\{x, y\}$ for all $x, y \in X$. Consider the self-mapping $T : X \to X$ given by

$$Tx = \begin{cases} 2x - \dfrac{7}{4} & if\ x > 1, \\ \dfrac{x}{4} & if\ 0 \leq x \leq 1. \end{cases}$$

Notice that 1.16, a characterization of the Bananch fixed point theorem, cannot be applied in this case because

$$d(T1, T2) = \frac{9}{4} > 2 = \mathrm{d}(1,2).$$

Define $\beta : X \times X \to [0, \infty)$, as

$$\beta(x, y) = \begin{cases} 1 & if\ x, y \in [0,1], \\ 0 & otherwise. \end{cases}$$

Let $\psi(t) = \frac{t}{2}$ for $t \geq 0$. Then we conclude that T is a d-β-ψ-contractive mapping. In fact, for all $x, y \in X$, we have

$$\beta(x, y) d(Tx, Ty) \leq \frac{1}{2} d(x, y).$$

On the other hand, there exists $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$. Indeed, for $x_0 = 1$, we have $\beta(1, T1) = \beta\left(1, \frac{1}{4}\right) = 1$

Notice also that T is continuous. To show that T satisfies all the hypotheses of Theorem 2.4, it is sufficient to observe that T is β-admissible. For this purpose, let $x, y \in X$ such that $\beta(x, y) \geq 1$, which is equivalent to saying that $x, y \in [0,1]$. due to the definitions of β and T, we have

$$Tx = \tfrac{x}{4} \in [0,1], \quad Ty = \tfrac{y}{4} \in [0,1].$$

Hence, $\beta(Tx, Ty) \geq 1$. As a result, all the conditions of theorem 2.4 are satisfied. Note that theorem 2.4 guarantees the existence of a fixed point but not the uniqueness. In this example 0 and $\frac{7}{4}$ are two fixed points of T.

In the following example T is not continuous.

Example 3.7: Let X, G and β be defined as in example 31. Let $T: X \to X$

$$Tx = \begin{cases} 2x - \dfrac{7}{4} & if\ x > 1, \\ \dfrac{x}{3} & if\ 0 \leq x \leq 1. \end{cases}$$

Let $\psi(t) = \frac{t}{3}$ for $t \geq 0$. then we conclude that T is a d-β-ψ-contractive mapping. In fact, for all $x, y \in X$, we have

$$\beta(x, y) d(Tx, Ty) \leq \frac{1}{2} d(x, y)$$

Furthermore, there exist $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$. For $x_0 = 1$, we have $\beta(1, T1) = \beta\left(1, \frac{1}{3}\right) = 1$.

Let $\{x_n\}$ be a sequence such that $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and as $x_n \to x$ as $n \to \infty$. by the definition of β, we have $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. then we see that $x_n \in [0,1]$. Thus, $\beta(x_n, x) \geq 1$.

To show that T satisfies all the hypotheses of Theorem 2.5, it is sufficient to observe that T is β –admissible. For this purpose let$x,\ y \in X$ such that $\beta(x,y) \geq 1$. It is equivalent to saying that $x, y \in [0,1]$. Due to the definition of β and T, we have

$$Tx = \frac{x}{3} \in [0,1], \qquad Ty = \frac{y}{3}[0,1].$$

Hence $\beta(Tx, Ty) \geq 1$.

As a result, all the conditions of the theorem 2.5 are satisfied. In this example, 0 and $\frac{7}{4}$ are two fixed points of T.

Theorem 3.8: Adding the following condition to the hypotheses of theorem 2.4 and 2.5 we obtain the uniqueness of a fixed point of T.

(i) For all $x, y \in X$, there exist $z \in X$ such that $\beta(x,z) \geq 1$ and $\beta(y,z) \geq 1$
$\beta(u,z) \geq 1 \qquad \beta(u^*,z) \geq 1$

Since T is β-admissible, we get by induction that

$$\beta(u, T^n z) \geq 1 \qquad \beta(u^*, T^n z) \geq 1 \qquad \text{for all } n = 1, 2, 3, \ldots \tag{6}$$

From (9) and (5), we have

$$\begin{aligned} d(u, T^n z) &= d(Tu, T(T^{n-1}z)) \\ &\leq \beta(u, T^{n-1}z) d(Tu, T(T^{n-1}z)) \\ &\leq \psi\big(d(u, T^{n-1}z)\big) \end{aligned}$$

Thus, we get by induction that

$$d(u, T^n z) \leq \psi^n\big(d(u,z)\big) \quad \text{for all } n = 1,2,3, \ldots$$

Letting $n \to \infty$, and since $\psi \in \Psi$, we have

$$d(u, T^n z) \to 0.$$

This implies that $\{T^n z\}$ is d-convergent to u. Similarly, we get $\{T^n z\}$ is d-convergent to u^*. By the uniqueness of the limit, we get $u = u^*$, that is, the fixed point of T is unique.

4. CYCLIC CONTRACTION

Now, we prove our results for cyclic contractive mappings in a d-metric space.

Theorem 4.1: Let A, B be a non-empty d-closed subset of a complete d-metric (X, d) space, let$Y = A \cup B$, and let $T\colon Y \to Y$ be a given self-mapping satisfying

$$T(A) \subset B \text{ and } T(B) \subset A. \tag{7}$$

If there exists a function $\psi \in \Psi$ such that

$$d(Tx, Ty) \leq \psi(d(x,y)) \text{ for all } x \in A, x \in B \tag{8}$$

Then T has a unique fixed point $u \in A \cap B$, that is,$Tu = u$.

Proof: Notice that (Y, d) is a complete d-metric space because A, B are closed subsets of a complete d-metric space (X, d).

We define $\beta\colon X \times X \to [0, \infty)$ in the following way:

$$\beta(x,y) = \begin{cases} 1 & if\ (x,y) \in (A \times B) \cup (B \times A) \\ 0 & otherwise \end{cases}$$

Due to the definition of β and (9), we have

$$\beta(x,y) d(Tx, Ty) \leq \psi\big(d(x,y)\big) \text{ for all } x, y \in X \tag{9}$$

Hence, T is a d-β-ψ-contractive mapping.

Let$(x, y) \in (Y \times Y)$ such that $\beta(x, y) \geq 1$.

If $(x, y) \in A \times B$, then by (8), $(Tx, Ty) \in B \times A$ which yields that $\beta(Tx, Ty) \geq 1$.

If $(x, y) \in B \times A$ we get again $\beta(Tx, Ty) \geq 1$ by analogy.

Thus, in any case we have $\beta(Tx, Ty) \geq 1$, that is, T is β-admissible. Notice also that for any $z \in A$ we have (z, Tz), which yields that$\beta(z, Tz) \geq 1$.

Take a sequence $\{x_n\}$ in X such that $\beta(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to z \in X$ as$n \to \infty$. Regarding the definition of β, we derive that

$$(x_n, x_{n+1}) \in (A \times B) \cup (B \times A) \text{ for all } n. \tag{10}$$

By the assumption, A, B and$(A \times B) \times (B \times A)$ are closed sets. Hence we get that$(z, z) \in (A \times B) \cup (B \times A)$, which implies that$z \in A \cap B$. We conclude, by the definition of β, that $\beta(x_n, z) \geq 1$ for all n.

Now all the hypotheses of Theorem 2.5 are satisfied, and we conclude that T has a fixed point. Next, we show the uniqueness of a fixed point z of T. Suppose that $w = Tw$, where $w \in A \cap B$. Since$(z, w) \in (A \times B) \cup (B \times A)$, we have $\beta(y, z) \geq 1$ and $\beta(z, y) \geq 1$. Thus the condition (iv) of Theorem 2.8 is satisfied.

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Source of support: Nil, Conflict of interest: None Declared