International Journal of Mathematical Archive-7(9), 2016, 10-13
Available online through www.ijma.info ISSN 2229 – 5046

EXISTENCE OF SOLUTION OF RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING THE SUM OF TWO FUNCTIONS

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(Received On: 09-08-16; Revised & Accepted On: 07-09-16)

ABSTRACT

In this paper, the method of upper and lower solution together with the monotone iterative technique for the Riemann-Liouville Fractional initial value problem (in short FIVP) involving the sum of two functions is developed.

Keywords: *Fractional initial value problem, coupled lower and upper solutions, existence of solution.*

1. INTRODUCTION

The concept of non-integer order derivatives, popularly known as fractional derivative goes back to the seventeenth century [1, 2]. Since that time the fractional calculus has drawn the attention of many famous mathematicians. It is only a few decades ago, it was realized that the derivatives of arbitrary order provide an excellent framework for modeling the real world problems in a variety of disciplines. There has been a growing interest in this new area to study the concept of fractional differential equations and fractional dynamical systems [3, 4, 5]. Many methods are available to study different aspects of solutions such as Power series method, adomain method, quasilinerazation method, Transform method.

The method of upper and lower solutions coupled with the monotone iterative technique is effective tool that offers theoretical as well as constructive existence results in a closed set that is generated by upper and lower solutions see[6]. Development in the theory and applications of fractional differential equations is growing day by day. The basic theory of fractional differential equations has been developed by many researchers see [7, 8]. In 2004, I.H. West and A.S.Vatsala [9] was developed generalized monotone iterative techniques for nonlinear problems involving the difference of two monotone functions. J.A. Nanware [11] obtain existence result for nonlinear initial value problems involving the difference of two monotone functions.

Monotone iterative method for the initial value problem

$$u' = f(t,u(t)) + g(t,u(t)), \quad u(0) = u_0 \quad on\ J = [0,T], T > 0$$

Where $f, g \epsilon\, C[\mathrm{J}\ X\ \mathbb{R}, \mathbb{R}]$ and $f(t,u(t))$ nondecreasing in u and $g(t,u(t))$ is nonincreasing in u on J, is developed by West and Vatsala [9]. In this paper, monotone iterative method together with upper and lower solution is developed to obtain existence of solution of fractional differential equations.

2. BASIC DEFINITIONS AND AUXILLARY RESULT

Consider the following Riemann-Liouville fractional differential equation

$$\mathrm{D}_t^q u(t) = f(t,u(t)) + g(t,u(t)), \quad on\ J = [0,T], T > 0 \tag{2.1}$$

$$u(0) = u_0 \tag{2.2}$$

Where $f, g \epsilon\, C[\mathrm{J}\ X\ \mathbb{R}, \mathbb{R}]$ and $f(t,u(t))$ nondecreasing in u and $g(t,u(t))$ is nonincreasing in u on J.

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Definition 2.1[5]: The Riemann-Liouville fractional derivative of order q (0 < q < 1) is defined as

$$\mathrm{D}_t^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^q} ds \qquad (2.3)$$

Definition 2.2[5]: The Riemann-Liouville fractional integral of order q (0 < q < 1) is defined as

$$\mathrm{I}_t^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds \qquad (2.4)$$

Definition 2.3: A pair of functions $\alpha(t)$ and $\beta(t)$ in $C_p(J, \mathbb{R})$ are called natural lower and upper solutions of the nonlinear IVP (2.1)-(2.2) if

$$\mathrm{D}_t^q \alpha(t) \le f\big(t, \alpha(t)\big) + g\big(t, \alpha(t)\big), \qquad \alpha(0) \le u_0 \text{ on J}$$

$$\mathrm{D}_t^q \beta(t) \ge f\big(t, \beta(t)\big) + g\big(t, \beta(t)\big), \qquad \beta(0) \ge u_0 \text{ on J}$$

Definition 2.4: A pair of functions $\alpha(t)$ and $\beta(t)$ in $C_p(J, \mathbb{R})$ are called coupled lower and upper solutions of the nonlinear IVP (2.1)-(2.2) if

$$\mathrm{D}_t^q \alpha(t) \le f\big(t, \alpha(t)\big) + g\big(t, \beta(t)\big), \qquad \alpha(0) \le u_0 \text{ on J}$$

$$\mathrm{D}_t^q \beta(t) \ge f\big(t, \beta(t)\big) + g\big(t, \alpha(t)\big), \qquad \beta(0) \ge u_0 \text{ on J}$$

Lemma 2.1 [**12**]: Let $m \epsilon C_p(J, \mathbb{R})$ be locally Holder continuous such that for any $t_1 \epsilon (t_0, T)$, one has $m(t_1) = 0$ and $m(t) \le 0$ *or* $m(t) \ge 0$ for $t_0 \le t \le t_1$. Then it follows that $\mathrm{D}^q m(t_1) \ge 0$ or $\mathrm{D}^q m(t_1) \le 0$, respectively.

Theorem 2.1 [14]: Let $\alpha, \beta \ \epsilon \ C_p(J, \mathbb{R})$, $f \epsilon \ C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and:

(i) $\mathrm{D}_t^q \alpha(t) \le f\big(t, \alpha(t)\big)$

and: (ii) $\mathrm{D}_t^q \beta(t) \ge f\big(t, \beta(t)\big)$ $t_0 < t \le$ T. Assume $f\big(t, u(t)\big)$ satisfies the Lipschitz condition

$$f(t,x) - f(t,y) \le L(x-y), x \le y, L > 0$$

Then $\alpha_0 < \beta_0$, where $\alpha^0 = \alpha(t)(t-t_0)^{1-q}]t = t_0$ *and* $\beta^0 = \beta(t)(t-t_0)^{1-q}]t = t_0$ implies that

$$\alpha(t) \le \beta(t), t \epsilon [t_0, T].$$

3. MAIN RESULTS

In this section, we develop monotone iterative scheme for fractional differential equation of IVP (2.1)-(2.2).

Theorem 3.1: Assume that

(i) $\alpha_0, \beta_0 \ \epsilon \ C[J, \mathbb{R}]$ are the coupled lower and upper solutions of IVP (2.1)-(2.2) with $\alpha_0 \le \beta_0$, on J.

(ii) $f, g \ \epsilon [J \times \mathbb{R}, \mathbb{R}]$, $f\big(t, u(t)\big)$ is nondecreasing in u on J and $g\big(t, u(t)\big)$ is nondecreasing in u on J.

Then their exists monotone sequences $\alpha_n(t)$ *and* $\beta_n(t)$ *on* J such that $\alpha_n(t) \to \alpha(t)$ *and* $\beta_n(t) \to \beta(t)$ uniformly and monotonically on J and $\big(\alpha(t), \beta(t)\big)$ are coupled minimal and maximal solutions, respectively, to equations (2.1)-(2.2).

Proof: In order that $\big(\alpha(t), \beta(t)\big)$ be coupled minimal and maximal solutions, respectively, to equation (2.1)-(2.2) they must satisfy

$$\mathrm{D}_t^q \alpha(t) = f\big(t, \alpha(t)\big) + g\big(t, \beta(t)\big), \qquad \alpha(0) = u_0 \text{ on J} \qquad (3.1)$$

$$\mathrm{D}_t^q \beta(t) = f\big(t, \beta(t)\big) + g\big(t, \alpha(t)\big), \qquad \beta(0) = u_0 \text{ on J} \qquad (3.2)$$

Where the iterative scheme is given by

$$\mathrm{D}_t^q \alpha_{n+1}(t) = f\big(t, \alpha_n(t)\big) + g\big(t, \beta_n(t)\big), \ \alpha_{n+1}(0) = u_0 \text{ on J} \qquad (3.3)$$

$$\mathrm{D}_t^q \beta_{n+1}(t) = f\big(t, \beta_n(t)\big) + g\big(t, \alpha_n(t)\big), \ \beta_{n+1}(0) = u_0 \text{ on J} \qquad (3.4)$$

It is easy to see that the solutions of the linear initial value problem (3.3)-(3.4) exists and are unique for each k =1, 2,… we will prove that $\alpha_k, \beta_k \epsilon [\alpha_0, \beta_0] = \Omega = [u \ \epsilon \ C(J, \mathbb{R}): \alpha_0(t) \le u \le \beta_0(t), t \epsilon J]$, *with* $\alpha_k \le \beta_k$ for each $k \ge 1$.

Our aim is to show that

$$\alpha_0 \le \alpha_1 \le \alpha_2 \ldots \alpha_k \le \beta_k \le \cdots \le \beta_2 \le \beta_1 \le \beta_0 \qquad (3.5)$$

On J.

We claim that $\alpha_0 \le \alpha_1$ *and* $\beta_0 \ge \beta_1$. For this porpose, let $p(t) = \alpha_0 - \alpha_1$, *then*

$$\begin{aligned} \mathrm{D}_t^q p(t) &= \mathrm{D}_t^q \alpha_0(t) - \mathrm{D}_t^q \alpha_1(t) \\ &\le f\big(t, \alpha_0(t)\big) + g\big(t, \beta_0(t)\big) - f\big(t, \alpha_0(t)\big) - g\big(t, \beta_0(t)\big) \le 0 \end{aligned}$$

and $\quad p(0) = \alpha_0(0) - \alpha_1(0) \le u_0 - u_0 = 0$

$p(0) \le 0$ *It follows that* $p(t) \le 0$ *on* J.

This proves that $\alpha_0(t) \le \alpha_1(t)$ *on* J. Similarly, we can show that $\beta_0 \ge \beta_1$.

Furthermore we claim that $\alpha_1 \le \beta_1$. For that purpose set $p(t) = \alpha_1 - \beta_1$ then

$$\begin{aligned} \mathrm{D}_t^q p(t) &= \mathrm{D}_t^q \alpha_1(t) - \mathrm{D}_t^q \beta_1(t) \\ &\le f\big(t, \alpha_0(t)\big) + g\big(t, \beta_0(t)\big) - f\big(t, \beta_0(t)\big) - g\big(t, \alpha_0(t)\big) \le 0 \end{aligned}$$

Using monotone nature of f *and* g *and the fact that* $\alpha_0 \le \beta_0$ *It follows that* $p(t) \le 0$ *on* J.

This proves that $\alpha_1(t) \le \beta_1(t)$ *on* J since $p(0) = 0$. Thus, we have shown that

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t)$$

holds on J. Hence (3.5) is true for k = 1.

Now we assume that (3.5) holds for some k$\ge$ 1 on J. Then all we need to show that (3.5) holds for k+1. Thus, we need to show that

$$\alpha_k(t) \le \alpha_{k+1}(t) \le \beta_{k+1}(t) \le \beta_k(t)$$

holds on J

For this purpose, let $p(t) = \alpha_k(t) - \alpha_{k+1}(t)$ and note that $\alpha_k(0) - \alpha_{k+1}(0) = 0$, we get

$$\begin{aligned} \mathrm{D}_t^q p(t) &= \mathrm{D}_t^q \alpha_k(t) - \mathrm{D}_t^q \alpha_{k+1}(t) \\ &\le f\big(t, \alpha_{k-1}(t)\big) + g\big(t, \beta_{k-1}(t)\big) - f\big(t, \alpha_k(t)\big) - g\big(t, \beta_k(t)\big) \le 0 \end{aligned}$$

Using monotone nature of f *and* g. This proves that $\alpha_k(t) \le \alpha_{k+1}(t)$.

Similarly, we can prove that $\beta_{k+1}(t) \le \beta_k(t)$. Now, we need to prove that $\alpha_{k+1}(t) \le \beta_{k+1}(t)$ on J

Consider $p(t) = \alpha_{k+1}(t) - \beta_{k+1}(t)$ and note that $\mathrm{p}(0) = 0$.

Then we get that

$$\begin{aligned} \mathrm{D}_t^q p(t) &= \mathrm{D}_t^q \alpha_{k+1}(t) - \mathrm{D}_t^q \beta_{k+1}(t) \\ &\le f\big(t, \alpha_k(t)\big) + g\big(t, \beta_k(t)\big) - f\big(t, \beta_k(t)\big) - g\big(t, \alpha_k(t)\big) \le 0 \end{aligned}$$

Using monotone nature of f *and* g. This proves that (3.5)holds for $\mathrm{k} + 1$. Hence (3.5) is valid for all $\mathrm{k} = 1,2, \ldots$

Also, the sequences $\alpha_k(t)$and $\beta_k(t)$ can be shown to be equicontinuous and uniformly bounded. Thus by Ascoli-Arzela's theorem, subsequences $\alpha_k(t)$and $\beta_k(t)$ converges to $\alpha(t)$ *and* $\beta(t)$ respectively on J. Since the sequences $\alpha_k(t), \beta_k(t)$ are monotone, the entire sequences converge uniformly and monotonically to $\alpha(t)$ *and* $\beta(t)$, respectively, on J. Therefore, $\alpha(t)$ *and* $\beta(t)$ satisfy the FIVP (2.1)-(2.2).

Using corresponding Volterra integral equations

$$\alpha_{n+1}(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{ f\big(t, \alpha_n(s)\big) + g\big(t,, \beta_n(s)\big) \} ds \tag{3.6}$$

$$\beta_{n+1}(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{ f\big(t, \beta_n(s)\big) + g\big(t, \alpha_n(s)\big) \} ds$$

It follows that $\alpha(t)$ *and* $\beta(t)$are solutions of (2.1)- (2.2) .

Finally, we claim that α *and* β are coupled minimal and maximal solutions of (2.1)-(2.2). Suppose that u is any solution of (2.1)-(2.2) such that $\alpha_0(t) \le u(t) \le \beta_0(t)$ *on* J. Then we will prove that

$$\alpha_0(t) \le \alpha(t) \le u(t) \le \beta(t) \le \beta_0(t) \text{ on } J.$$

First we will show that

$$\alpha_k(t) \le u(t) \le \beta_k(t) \tag{3.7}$$

holds for any $\mathrm{k} \ge 1$ on J.

We need to show that (3.7) is true for k=1. For this purpose, let $p(t) = \alpha_1 - u$, we get

$$\begin{aligned} D_t^q p(t) &= D_t^q \alpha_1(t) - D_t^q u\,(t) \\ &\le f\big(t, \alpha_0(t)\big) + g\big(t, \beta_0(t)\big) - f\big(t, u(t)\big) - g\big(t, u(t)\big) \le 0 \end{aligned}$$

Which implies that $\alpha_1 \le u,$ similarly we can show that $\beta_0 \ge u$. This proves that (3.7) holds for k=1. We will assume now that (3.7) is true for some $k > 1$ and the monotone nature of f *and* g.

Since $p(0) = 0$ on J, it follows that $p(t) \le 0$, which proves that

$$\alpha_{k+1}(t) \le u(t) \le \beta_k(t)$$

holds for any $k \ge 1$ on J. Now taking limit as $k \to \infty$, *we get* $\alpha(t) \le u(t) \le \beta(t)$ *on* J.

This completes the proof.

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Source of support: Nil, Conflict of interest: None Declared