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COMMON FIXED POINT THEOREM IN FUZZY METRIC SPACE

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ABSTRACT

In this paper, we prove a fixed point theorem for fuzzy metric space using five mappings for ε-chainable.our paper extended and generalized.

***Keywords**: ε-chainable, fuzzy metric space, weakly compatible, semi – compatible maps.*

***Mathematics Subject Classification:** 47H10, 54H25.*

1. INTRODUCTION

Zadeh development of mathematics when the notion of fuzzy set. which laid the foundation of fuzzy mathematics. Consequently the last three decades were very productive for fuzzy mathematics and the recent literature has observed the fuzzification in almost every direction of mathematics such as arithmetic, topology, graph theory, probability theory, logic etc. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. No wonder that fuzzy fixed point theory has become an area of interest for specialists in fixed point theory, or fuzzy mathematics has offered new possibilities for fixed point theorists.

Deng [4], Erceg [5], Kaleva and Seikkala [11] and Kramosil and Michalek [12] have introduced the concept of fuzzy metric spaces in various ways. George and Veeramani [8] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [12] and defined Hausdorff topology of metric spaces which is later proved to be metrizable. every metric induces a fuzzy metric Recently, Chugh and Kumar [3] proved a Pant type theorem for two pairs of R-weakly commutingma ppings satisfyinga Boyd and Wong [1] type contraction condition which in turn, generalizes a fixed point theorem of Vasuki [18]

2. PRELIMINARIES

Definition 2.1 (cf. [20]): A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2.2 (cf. [16]): A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if $\{[0, 1], *\}$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

Definition 2.3 (cf. [12]): The triplet (X, M, $*$) is a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions:

(i) $M(x, y, 0) = 0$,
(ii) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
(iii) $M(x, y, t) = M(y, x, t) \neq 0$ for $t \neq 0$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y, .) : [0,\infty) \rightarrow [0, 1]$ is left continuous for all $x, y, z \in X$ and $s, t > 0$.

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Example 2.1 (cf. [8]): Every metric space induces a fuzzy metric space. Let (X, d) be a metric space. Define a * b = ab and $M(x,y,t)=\dfrac{kt^n}{kt^n+md(x,y)}$, k, m, n, t$\in \Re^+$. Then (X, M, *) is a fuzzy space. If we put k = m = n = 1, we get

$$M(x,y,t)=\frac{t}{t+d(x,y)}$$

The fuzzy metric induced by a metric d is referred to as a standard fuzzy metric.

Definition 2.4 (cf. [10]): A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is convergent to x ∈ X if $\lim_{n\to\infty} m(x_n,x,t)=1$ for each t > 0.

Recently, Song [17] and Vasuki and Veeramani [19] again critically reviewed the existing definitions of Cauchy sequence in a fuzzy metric space. Vasuki and Veeramani [19] suggested that the definition of Cauchy sequence due to Grabiec [10] is weaker than that contained in [17, 19] and called it a G-Cauchy sequence.

Definition 2.5 (cf. [10]): A sequence $\{x_n\}$ in a fuzzy metric space (X,M,*) is called Cauchy if $\lim_{n\to\infty} M(x_n+x_n,t)=1$ for every t > 0 and each p > 0. (X, M,*) is complete if every Cauchy sequence in X converges in X.

Definition 2.6: A pair of self-mappings (f, g) of a fuzzy metric space (X, M, *) is said to be
(i) weakly commuting (cf.[18]) if M(fgx, gfx, t) ≥ M(fx, gx, t),
(ii) R-weakly commuting (cf. [18]) if there exists some R > 0 such that M(fgx, gfx, t) ≥ M(fx, gx, t/R),
(iii) R-weakly commuting mappings of type (Af) if there exists some R > 0 such that M(fgx, ggx, t) ≥ M(fx, gx, t/R),
(iv) R-weakly commuting mappings of type (Ag) if there exists some R > 0 such that M(gfx, ffx, t) ≥ M(fx, gx, t/R),
(v) R-weakly commuting mappings of type (P) if there exists some R > 0 such that M(ffx, ggx, t) ≥ M(fx, gx, t/R),
for all x ∈ X and t > 0.

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Example 2.2 (cf. [18]): Let $X=\Re$, the set of real numbers. Define a * b = ab and

$$M(x,y,t)=\begin{cases}\left(e^{\frac{|x-y|}{t}}\right)^{-1}, & for\ all\ x,y\in X\ and\ t>0\\ 0, & for\ all\ x,y\in X\ and\ t=0\end{cases}$$

Then it is well known (cf. [18]) that (X, M, *) is a fuzzy metric space. Define fx = 2x − 1 and gx = x^2. Then by a straightforward calculation, one can show that

$$M(fgx,gfx,t)=\left(e^{\frac{2|x-1|^2}{t}}\right)=M(fx,gx,t/2)$$

which shows that the pair (f, g) is R-weakly commuting for R=2. Note that the pair (f, g) is not weakly commuting due to a strict increasing property of the exponential function.

However, various kinds of above mentioned 'R-weak commutativity' notions are independent of one another and none implies the other. The earlier example can be utilized to demonstrate this inter-independence.

To demonstrate the independence of 'R-weak commutativity' with 'R-weak commutativity' of type (Af) notice that

$$M(fgx,ggx,t)=\left(e^{\frac{|x^4-2x^2+1|}{t}}\right)^{-1}=\left(e^{\frac{R(x-1)^2}{t}\frac{(x+1)^2}{R}}\right)^{-1}$$
$$<\left(e^{\frac{R|x-1|^2}{t}}\right)^{-1}=M(fx,gx,t/R) \text{ when } x>1$$

which shows that 'R-weak commutativity' does not imply 'R-weak commutativity' of type (Af).

Secondly, in order to demonstrate the independence of 'R-weak commutativity' with 'R-weak commutativity' of type (P) note that

$$M(ffx, ggx, t) = \left(e^{\frac{|x^4-4x+3|}{t}}\right)^{-1} = \left(e^{\frac{R(x-1)^2}{t}\frac{(x^2+2x+3)^2}{R}}\right)^{-1}$$

$$< \left(e^{\frac{R|x-1|^2}{t}}\right)^{-1} = M(fx, gx, t/R) \text{ for } x > 1.$$

Finally, for a change the pair (f, g) is R-weakly commutingof type (Ag) as

$$M(gfx,\ ffx, t) = \left(e^{\frac{|(2x-1)^2-4x+3|}{t}}\right)^{-1} = \left(e^{\frac{4|x-1|^2}{t}}\right)^{-1}$$

= M(fx, gx t/4)

which shows that (f, g) is R-weakly commuting of type (Ag) for R=4. This situation may also be utilized to interpret that an R-weakly commuting pair of type (Ag) need not be R-weakly commuting pair of type (Af) or type (P)

Theorem 3.1: Let A, B, S, T, L and M be a complete ε-chainable fuzzy metric space (X, M *) with continuous t-norm satisfying the conditions.

(1) $L(X) \subseteq ST(X),\ M(X) \subseteq AB(X)$

(2) AB = BA, ST = TS, LB = BL, MT = TM;

(3) M is ST-absorbing;

(4) there exists k ∈ (0, 1) such that

$$\min\{M(Lx, My, kt)\{M(ABx, My, 2t), M(ABx, STy, t), M(ABx, Lx, t), M(STy, My, t)\} \geq 0$$

For every x, y ∈ X, α ∈ (0, 2) and t > 0. If L, AB is reciprocally continuous, semi-compatible maps. Then A, B, S, T, L and M have a unique common fixed point in X.

Proof: Let $x_0 \in X$ then from (1) there exists $x_1, x_2 \in X$ such that $Lx_0 = STx_1 = y_0$ and $Mx_1 = ABx_2 = y_1$. In general we can find a sequence $\{x_n\}$ and $\{y_n\}$ in X such that $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Mx_{2n+1}\ y = x_{2n+1}$ for t > 0 and α = 1 – q with q ∈ (0, 1) in (4), we have

$$M(y_{2n+1}, y_{2n+2}, kt) = M(Lx_{2n+2}, Mx_{2n+1}, kt)$$

$$\min\left\{\begin{matrix} M(ABx_{2n+2}, Mx_{2n+1}, (2-(1-q)t)), M(ABx_{2n+2}, STx_{2n+1}, t) \\ M(ABx_{2n+2}, Lx_{2n+2}, t), M(STx_{2n+1}, Mx_{2n+1}, t) \end{matrix}\right\} \geq 0$$

$$\min\left\{\begin{matrix} M(y_{2n+1}, y_{2n+1}, ((1+q)t)), M(y_{2n+1}, y_{2n}, t) \\ M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+1}, t) \end{matrix}\right\} \geq 0$$

$$= \min\{1, M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t)\}$$

$$M(y_{2n+1}, y_{2n+2}, kt) \geq \min M(y_{2n+1}, y_{2n}, t)$$

Again $x = x_{2n+2}$ and $y = x_{2n+3}$ with α = 1-q with q ∈ (0, 1) in (4), we have

$$M(y_{2n+2}, y_{2n+3}, kt) = M(Lxy_{2n+2}, Mx_{2n+3}, kt)$$

$$\min\left\{\begin{matrix} M(ABx_{2n+2}, Mx_{2n+3}, (1+q)t), M(ABx_{2n+2}, STx_{2n+3}, t) \\ M(ABx_{2n+2}, Lx_{2n+2}, t), M(STx_{2n+3}, Mx_{2n+3}, t) \end{matrix}\right\} \geq 0$$

$$\min\left\{\begin{matrix} M(y_{2n+1}, y_{2n+3}, ((1+q)t)), M(y_{2n+1}, y_{2n+2}, t) \\ M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n+3}, t) \end{matrix}\right\} \geq 0$$

$$= \min\left\{\begin{matrix} M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n+3}, qt), M(y_{2n+1}, y_{2n+2}, t), \\ M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n+3}, t) \end{matrix}\right\} \geq 0$$

As t-norm continuous, letting $q \to 1$ we have,

$$M(y_{2n+2}, y_{2n+3}, kt) \geq \min\{M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+3}, t)\}$$

Hence,

$$M(y_{2n+1}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t)$$

Therefore for all n; we have

$$M(y_n, y_{n+1}, t) \geq M(y_n, y_{n-1}, t/k) \geq M(y_n, y_{n-1}, t/k^2) \geq ..... \geq M(y_n, y_{n-1}, t/k^n) \to 1 \text{ as } n \to \infty.$$

For any $t > 0$. For each $\varepsilon > 0$ and each $t > 0$, we can choose $n_0 \,\varepsilon\, N$ such that $M(y_n, y_{n+1}, t) > 1 - \varepsilon$ for all $n > n_0$. For m, $n \,\varepsilon\, N$, we suppose $m \geq n$. Then we have that

$$M(y_n, y_m, t) \geq M(y_n, y_{n+1}, t/m-n), M(y_{n+1}, y_{n+2}, t, m-n),*.....$$
$$*M(y_{m-1}, y_m, t/m-n) > (1-\in)*(1-\in)*(1-\in)*.....*(1-\in) \geq (1-\in)$$

Hence $\{y_n\}$ is a Cauchy sequence in X; that is $y_n \to z$ in X; so its subsequences Lx_{2n}, STx_{2n+1}, ABx_{2n}, Mx_{2n+1} also converges to z. Since X is ε-chainable, there exists ε-chain from x_n to x_{n+1}, that is, there exists a finite sequence $x_n = y_1, y_2, ...y_t = x_{n+1}$ such that $M(y_i, y_{i-1}, t) > 1-\varepsilon$ for all $t > 0$ and $i = 1, 2, ...l$. Thus we have $M(x_n, x_{n+1}, t) > M(y_1, y_2, t/l) * M(y_2, y_3, t/l)* ...*M(y_{i-1}, y_i, t/l) > (1 - \varepsilon) * (1 - \varepsilon) * (1 - \varepsilon) *... * (1 - \varepsilon) * \geq (1 - \varepsilon)$, and so $\{x_n\}$ is a Cauchy sequence in X and hence there exists $z \in X$ such that $x_n \to z$. Since the pair of (L, AB) is reciprocal continuous; we have $\lim_{n\to\infty} L(AB)x_{2n} \to Lz$ and $\lim_{n\to\infty} AB(L)x_{2n} \to ABz$ and the semi compatibility of (L, AB) which gives $\lim_{n\to\infty} AB(L)x_{2n} \to ABz$, therefore $Lz = ABz$. We claim

$Lz = ABz = z.$

Step 1: Putting $x = z$ and $y = x_{2n+1}$ with $\alpha = 1$ in (4), we have

$$M(Lz, Mx_{2n+1}, kt) \geq \min\left\{\begin{array}{l} M(ABz, Mx_{2n+1}, t), M(ABz, STx_{2n+1}, t), \\ \qquad M(ABz, Lz, t), M(STx_{2n+1}, Mx_{2n+1}, t) \end{array}\right\}$$

Letting $n \to \infty$; we have

$$M(Lz, z, kt) \geq \min\{M(Lz, z, t), M(Lz, z, t), M(Lz, Lz, t), M(z, z, t)\}$$

i.e.

$z = Lz = ABz.$

Step 2: Putting $x = Bz$, $y = x_{2n+1}$ with $\alpha = 1$ in (4), we have

$$M(L(Bz), Mx_{2n+1}, kt) \geq \min\left\{\begin{array}{l} M(AB(Bz), Mx_{2n+1}, t), M(AB(Bz), STx_{2n+1}, t) \\ \qquad M(AB(Bz), L(Bz), t), M(STx_{2n+1}, Mx_{2n+1}, t) \end{array}\right\}$$

Since LB = BL, AB = BA, so L(Bz) = B(Lz) = Bz and AB(Bz) = B(ABz) = Bz letting $n \to \infty$; we have

$$M(Bz, z, kt) \geq \min\{M(Bz, z, t), M(Bz, z, t), M(Bz, z, t), M(z, z, t)\}$$

i.e.

$$M(bz, z, kt) \geq M(Bz, z, t)$$

Therefore

$L(X) \subseteq ST(X)$, there exists $u \in X$, such that z = Lz = Stu. Putting $x = x_{2n}$, $y = u$ with $\alpha = 1$ in (4), we have

$$M(Lx_{2n}, Mu, kt), \min\left\{\begin{array}{l} M(ABx_{2n}, Mu, 2t), M(ABx_{2n}, STu, t), \\ \qquad M(ABx_{2n}, Lx_{2n}, t), M(STu, Mu, t) \end{array}\right\} \geq 0$$

Letting $n \to \infty$; we have

$$M(z, Mu, kt), \min\{M(z, Mu, t), M(z, z, t), M(z, z, t), M(z, Mu, t)\} \geq 0$$

i.e.

$$M(z, Mu, kt) \geq M(z, Mu, t)$$

Therefore

Z = Mu = STu.

Since M is ST-absorbing; then

$$M(STu, STMu, kt) \geq M(STu, Mu, t/R) = 1$$

i.e.

STu = STMu => z = STz.

Step 4: Putting x = x_{2n}, y = z with α = 1 in (4), we have

$$M(Lx_{2n}, Mz, kt), \min\left\{\begin{array}{l} M(ABx_{2n}, Mz, t), M(ABx_{2n}, STz, t), \\ \qquad M(ABx_{2n}, Lx_{2n}, t), M(STz, Mz, t) \end{array}\right\} \geq 0$$

Letting n → ∞; we have

$$M(z, Mz, kt), \min\{M(z, Mz, t), M(z, z, t), M(z, z, t), M(z, Mz, t)\} \geq 0$$

i.e.

$$M(z, Mz, kt) \geq M(z, Mz, t)$$

Therefore

z = Mz = STz.

Step 5: Putting x = x_{2n}, y = Tz with α = 1 in (4), we have

$$M(Lx_{2n}, M(Tz), kt), \min\left\{\begin{array}{l} M(ABx_{2n}, M(Tz), t), M(ABx_{2n}, ST(Tz), t), \\ \qquad M(ABx_{2n}, Lx_{2n}, t), M(ST(Tz), M(Tz), t) \end{array}\right\} \geq 0$$

Since MT = TM, ST = TS therefore M(Tz) = T(Mz) = Tz, ST(Tz) = T(STz) = Tz;

Letting n → ∞; we have

$$M(z, Tz, kt), \min\{M(z, Tz, t), M(z, z, t), M(z, z, t), M(Tz, Tz, t)\} \geq 0$$

i.e.

$$M(z, Tz, kt) \geq M(z, Tz, t)$$

Therefore

z = Tz = Sz = Mz.

Hence

z = Az = Bz = Lz = Sz = Mz = Tz.

Uniqueness: Let w be another fixed point of A, B, L, S, M and T. Then putting x = u, y = w with α = 1 in (4) we have

$$M(Lu, Mw, kt), \min\left\{\begin{array}{l} M(ABu, Mw, t), M(ABu, STw, t), \\ \qquad M(ABu, Lu, t), M(STw, Mw, t) \end{array}\right\} \geq 0$$

$$\min\{M(u, w, t), M(u, w, t), M(u, u, t), M(w, w, t)\} \geq 0$$

Therefore

$$M(u, w, kt) \geq M(u, w, t)$$

Hence

z = w.

Corollary 3.2: Let A, B, S, T, L and M be a complete ε-chainable fuzzy metric space (X,M,*) with continuous t-norm satisfying the conditions (1) to (3) of theorem 3.1 and ; (5) there exists $k \in (0,1)$ such that

$$\min\left\{M(Lx, My, kt), \left\{\begin{array}{l} M(ABx, My, 2t), M(ABx, STy, t), \\ \qquad M(ABx, Lx, t), M(STy, My, t), M(STy, Lx, 2t) \end{array}\right\}\right\} \geq 0$$

For every x, y ∈ X, α ε (0, 2) and t > 0. If L, AB is reciprocally continuous, semi-compatible maps. Then A, b, S, T, L and M have a unique common fixed point in X.

REFERENCES

1. Y.J. Cho, B.K. Sharma, D.R. Sahu Semi-compatibility and fixed point Math. Apon. 42(1995), 1, 91-98.
2. S.H. Cho, J.H. Jung, On common fixed point theorems in fuzzy metric spaces, int. mathematical Forum, 1, 2006, 29, 144`-1451.
3. M.Erceg, Metric spaces in fuzzy set theory, J.Math. Anal. Appl., 69 (1979), 205-230.
4. M.Grabiec fixed points in fuzzy metric space fuzzy sets and system, 27, (1998) 385-389.
5. A.George and P.Veermani., On some results in fuzzy metric spaces. Fuzzy sets and sysem 64 (1994), 395-399.
6. G.Jungck, compatible mappings and fixed points, Internat. J.Math. Math. Sci. 9(4) (1986), 771-779.
7. O.Kramosil, J.Michelak, Fuzzy metric and statistical metric space, Kybernetika, 11 (1975), 326-334.
8. U.Mishra, A.S. Ranadive, & D.Gopal, Fixed point theorems via absorbing maps; Thai J.Math., vol. 6(1) (2008) 49-60.
9. A.S. Ranadive, A.P.Chouhan, Common fixed point theorem in fuzzy metric spaces, (Accepted) Proc. of Math. Soc., B.H.U. Vol. 24.
10. A.S.Ranadive, D.Gopa, U.Mishra, On some open problems of common fixed point theorems for a pair of non-compatible self-maps. Proc. Of Math. Soc., B.H.U. Vol. 20 (2004), 135-141.
11. B.Singh, and S.Jain, semi-compatibility and fixed point theorem in fuzzy metric space using implicit relation Int. Jour of Math. And Math. Sci. 2005: 16 (2005) 2617-2619.
12. B.Schwizer and Skalar, Statistical metric spaces, Pacific J.Math. 10 (1960), 313-334.
13. R.Vasuki, Common fixed point for r-weakly commuting maps in fuzzy metric spaces, Indian J.Pure and Applied Mathematice (1999), 419-423.
14. L.A. Zadeh, Fuzzy sets Inform. And Cont., 89 (1965) 338-353.

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