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NPQ-INJECTIVE MODULES

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ABSTRACT

Let M be a right R-module. A right R-module N is called nonessential principally M-injective (briefly, NPM-injective) if, for each nonessential principal submodule mR of M, any R-homomorphism from mR to N can be extended to an R-homomorphism from M to N. M is called nonessential principally quasi –injective (briefly, NPQ-injective) if, it is NPM-injective. In this paper, we give some characterizations and properties of NPQ-injective modules.

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1. INTRODUCTION

Let R be a ring. A right R-module M is called *principally injective* (or *$P-$injective*), if every $R-$homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$ where l and r are left and right annihilators, respectively. This notion was introduced by Camillo [2] for commutative rings.

In [7], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. Nicholson, Park, and Yousif [8] extended this notion of principally injective rings to the one for modules. In [5], W. Junchao introduced the definition of Jpc-injective rings, a ring R is called right Jpc-injective if for each $a \in R \setminus Z_r$, any R-homomorphism from aR to R can be extended to an R-homomorphism from R to R.

In this note we introduce the definition of NPQ-injective modules and give some characterizations and properties. Some results on principally quasi-injective modules [8] are extended to these modules.

Throughout this paper, R will be an associative ring with identity and all modules are unitary right $R-$modules. For right $R-$modules M and N, $\mathrm{Hom}_R(M,N)$ denotes the set of all $R-$homomorphisms from M to N and $S = \mathrm{End}_R(M)$ denotes the endomorphism ring of M. If X is a subset of M the right (resp. left) annihilator of X in R (resp. S) is denoted by $r_R(X)$ (resp. $l_S(X)$). By notation $N \subset^{\oplus} M$ $(N \subset^{e} M)$ we mean that N is a direct summand (an essential submodule) of M.

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2. NPM -INJECTIVE MODULES

Recall that a submodule K of a right $R-$module M is *essential* (or *large*) in M if, every nonzero submodule L of M, we have $K \cap L \neq 0$.

Definition 2.1: Let M be a right R-module. A right R-module N is called *nonessential principally* M *- injective* (briefly, NPM *-injective*) if, for each nonessential principal submodule mR of M, any R-homomorphism from mR to N can be extended to an R-homomorphism from M to N.

Example 2.2: Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field.

(1) Let $M_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ and $N_R = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. Then N is not M -injective but N is NPM -injective.

(2) If $M_R = R_R$ and $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, then N is NPM -injective.

Proof: (1) It is obvious that $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. For any R-homomorphism $\alpha : \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$ with $\alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$ for some $x \in F$, then $\alpha\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for every $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, so $\alpha = 0$. Therefore N is not M -injective.

We see that only $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a nonessential principal submodule of M, then N is NPM -injective.

(2) For $M_R = R_R$ and $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, so it is clear that only $X_1 = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ and $X_3 = N$ are nonzero proper nonessential principal submodules of M. Let $\varphi : X_1 \to N$ be an $R-$homomorphism. Since $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in X_1$, there exists $x_{11}, x_{12} \in F$ such that $\varphi\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}$.

Then
$$\varphi\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}.$$

It follows that $x_{11} = 0$.

Define $\hat{\varphi} : M \to N$ by $\hat{\varphi}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}$. It is clear that $\hat{\varphi}$ is an $R-$homomorphism.

Then $\hat{\varphi}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}$.

This show that $\hat{\varphi}$ is an extension of φ. By the similar proof of X_1, we can show for X_2 and it is clear for X_3. Then N is NPM -injective.

Lemma 2.3: Let M and N be right $R-$modules. Then N is NPM -injective if and only if for each nonessential principal submodule mR of M,

$$\mathrm{Hom}_R(M,N)m = l_N r_R(m).$$

Proof: Clearly, $\mathrm{Hom}_R(M,N)m \subset l_N r_R(m).$

Let $n \in l_N r_R(m).$ Define $\varphi: mR \to nR$ by $\varphi(mr) = nr$ for every $r \in R.$ Then φ is well-defined because $r_R(m) \subset r_R(n)$. It is clear that φ is an $R-$homomorphism. Since N is NPM-injective, there exists an $R-$homomorphism $\hat{\varphi}: M \to N$ such that $\hat{\varphi}\iota_1 = \iota_2\varphi,$ where $\iota_1 : mR \to M$ and $\iota_2 : nR \to N$ are the inclusion maps. Hence $n = \hat{\varphi}(m) \in \mathrm{Hom}_R(M,N)m.$

Conversely, let $m \in M$ with $mR \not\subset^e M$ and $\varphi: mR \to N$ be an $R-$homomorphism. Then $\varphi(m) \in l_N r_R(m)$ so by assumption, we have $\varphi(m) = \hat{\varphi}(m)$ for some $\hat{\varphi} \in \mathrm{Hom}_R(M,N).$ This shows that N is NPM -injective.

Lemma 2.4: Let N_i $(1 \le i \le n)$ be NPM -injective modules. Then $\oplus_{i=1}^n N_i$ is NPM -injective.

Proof: Let $m \in M$ with $mR \not\subset^e M$ and $\varphi: mR \to \oplus_{i=1}^n N_i$ be an $R-$homomorphism. Then for each i, there exists an $R-$homomorphism $\varphi_i : M \to N_i$ such that $\varphi_i \iota = \pi_i \varphi$ where $\pi_i : \oplus_{i=1}^n N_i \to N_i$ is the projection map, and $\iota: mR \to M$ is the inclusion map. Put $\hat{\varphi} = \iota_1\varphi_1 + \ldots + \iota_n\varphi_n : M \to \oplus_{i=1}^n N_i$. Then it is clear that $\hat{\varphi}$ extends φ.

Lemma 2.5:

(1) N is NPM -injective if and only if N is NPX -injective for any submodule X of M.

(2) Any direct summand of an NPM - injective module is again NPM - injective.

Proof: The sufficiency is trivial. For the necessity, let $m \in X$ with $mR \not\subset^e X$ and $\varphi: mR \to N$ be an $R-$homomorphism. Since $mR \not\subset^e M,$ there exists an $R-$homomorphism $\hat{\varphi}: M \to N$ such that $\varphi = \hat{\varphi}\iota_2\iota_1$ where $\iota_1 : mR \to X$ and $\iota_2 : X \to M$ are the inclusion maps. Then $\hat{\varphi}\iota_2$ extends φ.

(2) By definition.

Lemma 2.6: If $m \in M$ with $mR \not\subset^e M$ and mR is NPM -injective, then $mR \subset^{\oplus} M.$

Proof: Since mR is NPM -injective, there exists an $R-$homomorphism $\varphi: M \to mR$ such that $\varphi\iota = 1_{mR}$ where $\iota: mR \to M$ is the inclusion map. Then by [1, Lemma 5.1], ι is a split monomorphism, therefore $mR \subset^{\oplus} M.$

Theorem 2.7: The following conditions are equivalent for a projective module M.

(1) Every $m \in M$ with $mR \not\subset^e M,$ mR is projective.

(2) Every factor module of an NPM -injective module is NPM -injective.

(3) Every factor module of an injective $R-$module is NPM -injective.

Proof:

$(1) \Rightarrow (2)$ Let N be an NPM -injective, X a submodule of. N, $m \in M$ with $mR \not\subset^e M$ and let $\varphi: mR \to N/X$ be an $R-$homomorphism. Then by (1), there exists an $R-$homomorphism $\beta: mR \to N$ such that $\varphi = \eta\beta$ where $\eta: N \to N/X$ is the natural $R-$epimorphism. Since N is NPM -injective, there exists an $R-$homomorphism $\hat{\varphi}: M \to N$ which is an extension of β to M. Then $\eta\hat{\varphi}$ is an extension of φ to M.

$(2) \Rightarrow (3)$ is clear.

$(3) \Rightarrow (1)$ Let $m \in M$ with $mR \not\subset^e M,$ $h: A \to B$ an $R-$epimorphism, and let $\alpha: mR \to B$ be an $R-$homomorphism. Embed A in an injective module E [1, 18.6]. Let $\sigma: B \to A/\text{Ker}(h)$ be an $R-$isomorphism. Since $E/\text{Ker}(h)$ is NPM -injective, there exists an $R-$homomorphism $\hat{\alpha}: M \to E/\text{Ker}(h)$ such that $\iota_1\sigma\alpha = \hat{\alpha}\iota_2$ where $\iota_1 : A/\text{Ker}(h) \to E/\text{Ker}(h)$ and $\iota_2 : mR \to M$ are the inclusion maps.

Since M is projective, $\hat{\alpha}$ can be lifted to $\beta: M \to E.$ Let $x \in mR.$ Then $\sigma\alpha(x) = a + \text{Ker}(h)$ for some $a \in A,$ so $\beta(x) + \text{Ker}(h) = \eta\beta(x) = \hat{\alpha}(x) = \sigma\alpha(x) = a + \text{Ker}(h)$ where $\eta: E \to E/\text{Ker}(h)$ is the natural $R-$epimorphism. Hence $\beta(x) - a \in \text{Ker}(h) \subset A$ so $\beta(x) \in A.$ This shows that $\beta(mR) \subset A.$ Therefore we have lifted $\alpha.$

3. NPQ-INJECTIVE MODULES

A right R-module M is called *nonessential principally quasi –injective* (briefly, NPQ *-injective*) if, it is NPM -injective.

Lemma 3.1: Let M be a right $R-$module and $S = \text{End}_R(M).$ Then the following conditions are equivalent.

(1) M is NPQ-injective.

(2) $l_M r_R(m) = Sm$ for each $m \in M$ with $mR \not\subset^e M.$

(3) $r_R(m) \subset r_R(n),$ where $m, n \in M$ with $mR \not\subset^e M,$ implies that $Sn \subset Sm.$

(4) $l_M(r_R(m) \cap aR) = l_M(a) + Sm$ for all $a \in R$ and $m \in M$ with $maR \not\subset^e M.$

(5) If $\alpha: mR \to M$ is an $R-$homomorphism, $mR \not\subset^e M,$ then $\alpha(m) \in Sm.$

Proof:
$(1) \Leftrightarrow (2)$ by Lemma 2.3

$(2) \Rightarrow (3)$ If $r_R(m) \subset r_R(n),$ where $m, n \in M$ with $mR \not\subset^e M,$ then $l_M r_R(n) \subset l_M r_R(m).$ Then $Sn \subset l_M r_R(n) \subset l_M r_R(m) = Sm$ by (2).

$(3) \Rightarrow (4)$ Let $a \in R$ and $m \in M$ with $maR \not\subset^e M$ and let $x \in l_M(r_R(m) \cap aR).$
Then $r_R(ma) \subset r_R(xa),$ and hence by (3), $Sxa \subset Sma.$ Thus $xa = \varphi(ma),$ $\varphi \in S$ and so $(x - \varphi(m)) \in l_M(a).$ It follows that $x \in l_M(a) + Sm.$ The other hand is clear.

$(4) \Rightarrow (5)$ Put $a = 1_R$ in (4), then $\alpha(m) \in l_M r_R(m) = l_R(r_R(m) \cap 1R) = l_M(1_R) + Sm = Sm$ because $m1R \not\subset^e M.$

$(5) \Rightarrow (1)$ Let $m \in M$ with $mR \not\subset^e M$ and let $\varphi: mR \to M$ be an $R-$homomorphism.
Then by (5), $\varphi(m) \in Sm$ so there exists an $R-$homomorphism $\hat{\varphi} \in S$ is an extension of φ to $M.$
Following [8], a right $R-$module M is called a *principal self-generator,* if every element $m \in M$ has the form $m = \gamma(m_1)$ for some $\gamma: M \to mR.$ If $uR \neq 0$ is uniform, we call u a *uniform element* of $M.$ We call a right R-module M is a *duo module* if every submodule of M is fully invariant.

Theorem 3.2: Let M be a duo, NPQ -injective module and $m, n \in M$ with $mR \not\subset^e M.$

(1) If mR embeds into $nR,$ then Sm is an image of $Sn.$

(2) If nR is an image of $mR,$ then Sn can be embedded into $Sm.$

(3) If $mR \simeq nR,$ then $Sm \simeq Sn.$

Proof: (1) Let $\sigma : mR \to nR$ be an $R-$monomorphism and let $\iota_1 : mR \to M$ and $\iota_2 : nR \to M$ be the inclusion maps. Since M is NPQ-injective, there exists an $R-$homomorphism $\hat{\sigma} : M \to M$ such that $\hat{\sigma}\iota_1 = \iota_2\sigma.$ Let $\varphi : Sn \to Sm$ defined by $\varphi(\alpha(n)) = \alpha\hat{\sigma}(m)$ for every $\alpha \in S$. Since $\varphi(\alpha(n)) = \alpha(\hat{\sigma}(m)) = \alpha(\sigma(m)) \in \alpha(nR),$ φ is well-defined. It is clear that φ is an S-homomorphism. Since $\hat{\sigma}|_{mR}$ is monic and M is a duo module, $\hat{\sigma}(mR) \subset mR$ so $\sigma(mR) \not\subset^e M.$ Since $r_R(\sigma(m)) \subset r_R(m),$ $Sm \subset S\sigma(m)$ byLemma3.1Then $m \in S\sigma(m) \subset \varphi(Sn).$

(2) By the same notations as in (1), let $\sigma : mR \to nR$ be an $R-$epimorphism. Write $\sigma(ms) = n,$ $s \in R.$ Since M is NPQ-injective, σ can be extended to $\hat{\sigma} : M \to M$ such that $\hat{\sigma}\iota_1 = \iota_2\sigma.$ Define $\varphi : Sn \to Sm$ defined by $\varphi(\alpha(n)) = \alpha\hat{\sigma}(ms)$ for every $\alpha \in S$. It is clear that φ is an $S-$homomorphism. If $\alpha(n) \in Ker(\varphi)$, then $0 = \varphi(\alpha(n)) = \alpha\hat{\sigma}(ms) = \alpha(n).$ This shows that φ is an $S-$monomorphism.

(3) Follows from (1) and (2).

Theorem 3.3: Let M be a principal module which is a principal self-generator. Then the following conditions are equivalent.

(1) M is NPQ-injective.
(2) $l_S(Ker(\alpha) \cap mR) = l_S(m) + S\alpha$ for all $m \in M$ and $\alpha \in S$ with $\alpha(m)R \not\subset^e M.$
(3) $l_S(Ker(\alpha)) = S\alpha$ for all $\alpha \in S$ with $\alpha(M) \not\subset^e M.$
(4) $Ker(\alpha) \subset Ker(\beta),$ where $\alpha, \beta \in S$ with $\alpha(M) \not\subset^e M,$ implies that $S\beta \subset S\alpha.$

Proof: $(1) \Rightarrow (2)$ Clearly, $l_S(m) + S\alpha \subset l_S(Ker(\alpha) \cap mR).$ Let $\beta \in l_S(Ker(\alpha) \cap mR).$
Then $r_R(\alpha(m)) \subset r_R(\beta(m)),$ so $l_M r_R(\beta(m)) \subset l_M r_R(\alpha(m)).$ Since $\alpha(m)R \not\subset^e M,$
$S\beta(m) \subset l_M r_R(\beta(m)) \subset l_M r_R(\alpha(m)) = S\alpha(m)$ by Lemma 3.1, so $\beta(m) = \gamma\alpha(m)$ for some $\gamma \in S.$ It follows that $(\beta - \gamma\alpha) \in l_S(m),$ and hence $\beta \in l_S(m) + S\alpha.$

$(2) \Rightarrow (3)$ If $M = m_0R,$ take $m = m_0$ in (2).

$(3) \Rightarrow (4)$ $Ker(\alpha) \subset Ker(\beta),$ then $l_S(Ker(\beta)) \subset l_S(Ker(\alpha)).$ It follows that $S\beta \subset l_S(Ker(\beta)) \subset l_S(Ker(\alpha)) = S\alpha.$

$(4) \Rightarrow (1)$ Let $m \in M$ with $mR \not\subset^e M,$ $\varphi : mR \to M$ be an $R-$homomorphism.

Since M is a principal self-generator, there exists $\beta \in S$ such that $\beta(m_1) = m,$ so $Ker(\beta) \subset Ker(\varphi\beta)$ and $\beta(M) \not\subset^e M.$ Then by (4), $S\varphi\beta \subset S\beta$ hence $\varphi\beta = \hat{\varphi}\beta$ for some $\hat{\varphi} \in S.$ This shows that $\hat{\varphi}$ is an extension of $\varphi.$

Theorem 3.4: Let M be a duo, NPQ-injective module. If u is a *uniform element* of M with $uR \not\subset^e M,$ then $M_u = \{\alpha \in S \mid Ker(\alpha) \cap uR \neq 0\}$ is a unique maximal left ideal of S containing $l_S(u).$

Proof: Since uR is uniform, M_u is a left ideal of $S.$ It is clear that $l_S(u) \subset M_u \neq S.$
Let X be a left ideal of S containing $l_S(u)$ and $X \neq S.$ If $\alpha \in X - M_u$, then $Ker(\alpha) \cap uR = 0.$ Since M is a duo module, $\alpha(u)R \not\subset^e M$ and so by Theorem 3.3 we have $S = l_S(Ker(\alpha) \cap uR) = l_S(u) + S\alpha \subset X$ a contradiction. Thus $X \subset M_u$.

Definition 3.5: Let M be a right $R-$-module, $S = End_R(M)$. The module M is called *almost* NPQ*-injective* if, for each nonessential principal submodule mR of M, there exists an S-submodule X_m of M such that $l_M(r_R(m)) = Sm \oplus X_m$ as left S-modules.

Theorem 3.6: Let M be a right R-module, $S = End_R(M)$ and $m \in M$ with $mR \not\subset^e M$.

(1) If $Hom_R(mR, M) = S \oplus Y$ as left S-modules, then $l_M(r_R(m)) = Sm \oplus X$ as left S-modules, where $X = \{f(m) : f \in Y\}$.

(2) If $l_M(r_R(m)) = Sm \oplus X$ for some $X \subset M$ as left S-modules, then we have $Hom_R(mR, M) = S \oplus Y$ as left S-modules, where $Y = \{f \in Hom_R(mR, M) : f(m) \in X\}$.

(3) Sm is a direct summand of $l_M(r_R(m))$ as left S-modules if and only if S is a direct summand of $Hom_R(mR, M)$ as left S-modules.

Proof: Define $\theta : Hom_R(mR, M) \to l_M(r_R(m))$ by $\theta(f) = f(m)$ for every $f \in Hom_R(mR, M)$. It is obvious that θ is an S-monomorphism. For $x \in l_M(r_R(m))$, define $g : mR \to M$ by $g(mr) = xr$ for every $r \in R$. Since $r_R(m) \subset r_R(x)$, g is well-defined, so it is clear that g is an R-homomorphism. Then $\theta(g) = g(m) = x$. Therefore θ is an S-isomorphism. Let $\alpha(m) \in Sm$. Since $\alpha(m) \in l_M(r_R(m))$, there exists $\varphi \in Hom_R(mR, M)$ such that $\theta(\varphi) = \alpha(m)$, so $\varphi(m) = \alpha(m)$. Define $\hat{\varphi} : M \to M$ by $\hat{\varphi}(x) = \alpha(x)$ for every $x \in M$. It is clear that $\hat{\varphi}$ is an R-homomorphism and is an extension of φ. Then $\alpha(m) = \hat{\varphi}(m) = \theta(\hat{\varphi})$. This shows that $Sm \subset \theta(S)$. The other inclusion is clear. Then $\theta(S) = Sm$ and $X = \theta(Y) = \{f(m) : f \in Y\}$. Then the Lemma follows.

Theorem 3.7: The following conditions are equivalent:

(1) M is almost NPQ-injective.

(2) There exists an indexed set $\{X_m : m \in M\}$ of S-submodules of M with the property that if $mR \not\subset^e M$, $m \in M$, then $l_M(r_R(m) \cap aR) = (X_{ma} : a)_l + Sm$ and $(X_{ma} : a)_l \cap Sm \subset l_M(a)$ for all $a \in R$, where $(X_{ma} : a)_l = \{n \in M : na \in X_{ma}\}$ if $ma \neq 0$ and $(X_{ma} : a)_l = l_M(aR)$ if $ma = 0$.

Proof:

$(1) \Rightarrow (2)$ Let $m \in M$ with $mR \not\subset^e M$. Then there exists an S-submodule X_m of M such that $l_M(r_R(m)) = Sm \oplus X_m$ as left S-modules. Let $a \in R$. If $ma = 0$, then $aR \subset r_R(m)$ so (2) follows. If $ma \neq 0$, then any $x \in l_M(r_R(m) \cap aR)$ we have $r_R(ma) \subset r_R(xa)$ and so $xa \in l_M(r_R(xa)) \subset l_M(r_R(ma)) = Sma \oplus X_{ma}$ because $maR \not\subset^e M$. Write $xa = \alpha(ma) + y$ where $\alpha \in S$ and $y \in X_{ma}$. Then $(x - \alpha(m))a = y \in X_{ma}$, so $x - \alpha(m) \in (X_{ma} : a)_l$. It follows that $x \in (X_{ma} : a)_l + Sm$. This shows that $l_M(r_R(m) \cap aR) \subset (X_{ma} : a)_l + Sm$.

Conversely, it is clear that $Sm \subset l_M(r_R(m) \cap aR)$. Let $y \in (X_{ma} : a)_l$. Then $ya \in X_{ma} \subset l_M(r_R(ma))$. If $as \in r_R(m) \cap aR$, then $mas = 0$ and so $yas = 0$. Hence $y \in l_M(r_R(m) \cap aR)$. This shows that $(X_{ma} : a)_l \subset l_M(r_R(m) \cap aR)$. Therefore $l_M(r_R(m) \cap aR) = (X_{ma} : a)_l + Sm$.

If $\beta(m) \in (X_{ma} : a)_l \cap Sm$, then $\beta(m)a \in X_{ma} \cap Sma = 0$. Hence $\beta(m) \in l_M(a)$.

$(2) \Rightarrow (1)$ Let $m \in M$ with $mR \not\subset^e M$. Then there exists an S-submodule X_m of M such that $l_M(r_R(m)) = l_M(r_R(m) \cap R) = (X_m : 1)_l + Sm$ and $(X_m : 1)_l \cap Sm \subset l_M(1) = 0$.

Note that $(X_m : 1)_l = X_m$. Then (1) follows.

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