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MATRIX PRODUCT (modulo-2) OF PETERSEN GRAPHS

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ABSTRACT

Let G be simple graph of order n. $A(G)$ is the adjacency matrix of G of order $n \times n$. The matrix $A(G)$ is said to graphical if all its diagonal entries should be zero. The graph Γ is said to be the matrix product (mod-2) of G and $\bar{G}$ if $A(G)$ and $A(\bar{G})$(mod-2) is graphical and Γ is the realization of $A(G)\,A(\bar{G})$(mod-2). In this paper, we are going to study the realization of the Petersen graph G and any $k-$ regular subgraph of $\bar{G}$. Also some interesting characterizations and properties of the graphs for each the product of adjacency matrix under (mod-2) is graphical.

Keywords: *Adjacency matrix, Matrix product, Graphical matrix, Realization.*

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph. The order of G is the number of vertices of G. For any vertex $v \in V$ the open neighborhood of v is the set $N(v) = \{u \in V / uv \in E\}$ and the closed Neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = U_{v \in s}\, N(v)$ and the closed Neighborhood of $S\, is\, N[S] = N(S) \cup S$

A set $S \subseteq V$ is a dominating set if $N(S) = V - S,\ or$ equivalently, every vertex $in\, V/S$ is adjacent to at least one vertex in S.

Graphs considered in this paper are connected simple and undirected. Let G be any graph its vertices are denoted by $\{v_1, v_2, \dots, v_n\}$ two vertices $v_i\ and\ v_j, i \neq j$ are said to be adjacent to each other if there is an edge between them. An adjacency between the vertices v_i and v_j is denoted by $v_i \sim_G v_j\ and\ v_i \nsim_G v_j$ denotes that v_i is not adjacent with v_j in the graph G. The adjacency matrix of G is a Matrix $A(G) = (a_{ij}) \in M_n\,(R)\ in\ which\ a_{ij} = 1\ if\ \ v_i\ and\ v_j$ are adjacent, and $a_{ij} = 0$ otherwise, given two graphs G and H have the same set of vertices $\{v_1, v_2, \dots, v_n\}, G \cup H$ represents the union of graphs G and H having the same vertex set and two vertics are adjacent in $G \cup H$ if they are adjacent in at least one of G and H. Graphs G and H having the same set of vertices are said to be edge disjoint, if $u \sim_G v$ implies that $u \nsim_H v$ equivalently, H is a subgraph of G and G is a sub graph of H.

2. MATRIX PRODUCT (MODULO-2) OF PETERSEN GRAPHS

Definition: 2.1 Let G be a graph with n vertices, m edges, the incidence matrix A of G is an $n \times m$ matrix A=(a_{ij}), where n represents the number of rows correspond to the vertices and m represents the columns correspond to edges such that

$$(a_{ij}) = \begin{cases} 1 & if\ v_i\ and\ \ v_j\ are\ adjacent\ in\ G \\ 0 & otherwise \end{cases}$$

It is also called vertex-edge incidence matrix and is denoted by $\wedge(G)$.

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Definition: 2.2 A symmetric $(0,1)-$ Matrix is said to be graphical if all its diagonal entries equal Zero.

If B is a graphical matrix such that B=A(G) for some graph G, Then we often say that G is the realization of graphical matrix B.

Definition: 2.3 Let us Consider any two graphs G and H having same set of vertices. A graph Γ is said to be the matrix product of G and H. If $A(G)\,A(H)$ is graphical and Γ is the realization of $A(G)\,A(H)$. We shall extend the above definition of matrix product of graphs when the matrix multiplications is considered over the integers modulo-2.

Definition: 2.4 The graph Γ is said to be a matrix product (mod-2) of graphs G and $\bar{G}$ if $A(G)\,A(\bar{G})$(mod-2) is graphical and Γ is the realization of $A(G)\,A(\bar{G})$ (mod-2).

Definition: 2.5 Given graphs G and H on the same set of vertices $\{v_1, v_2, \dots v_n\}$, two vertices v_i and v_j $(i \neq j)$ are said to have a GH path if there exists a vertex v_k, different from v_i and v_j such that $v_i \sim_G v_k$ and $v_k \sim_H v_j$.

Definition: 2.6 A graph is a parity graph if for any two induced paths joining the same pair of vertices the path lengths have the same parity (odd or even).

Theorem: 2.7 Let G be Petersen graph and $\bar{G}$ be any k- regular sub graph of $\bar{G}$ $(k = 1,2,3,4,5,6)$. Then $A(G)\,A(\bar{G})$ is a graphical matrix.

Proof: Let $C_n = \{v_1, v_2, \dots, v_n\}$. v_i is adjacent with v_{i-1} and v_{i+1} such that $v_n = v_o$. Let (a_{ij}) is the adjacent matrix of G and (b_{ij}) is the adjacent matrix of $\bar{G}$.

Then, each

$$(a_{ij}) = \begin{cases} 1 & if\ j = i+1\ (or) j = i-1 \\ 0 & otherwise \end{cases} \quad \text{and}$$

$$(b_{ij}) = \begin{cases} 1 & if\ j = i+1\ (or) j = i-1 \\ 0 & otherwise \end{cases}$$

Then, $A(G)\,\mathrm{A}\overline{\mathrm{G}} = \{(c_{ij}) = 0\ if\ \ i = j, i = 1,2 \dots, n\ \}$

Hence all diagonal entries of $A(G)\,A(\bar{G})$ is zero. So Peterson graph is graphical matrix.

Theorem: 2.8 The diagonal entries of the matrix product $A(G)\,A(H\ )$ are zeros if and only if for each vertex $v_i \in G$ the cardinality of the set of vertices $\{v_k : v_k \sim_G v_i, v_k \sim_H v_i\}$ is even.

Proof: Let $A(G) = (a_{ij})$; $A(H\ ) = (b_{ij})$

$A(G)\,A(H\ ) = (c_{ij}) i = 1,2, \dots, 10$; $j = 1,2, \dots, 10$ [Since the adjacency matrices are symmetric]

We have $b_{kj} = b_{jk}\,; c_{ij} = \sum_k a_{ik}\, b_{kj}$

$= \sum_k a_{ik}\, b_{jk}$ (mod-2)

Taking $i = j$, we get that $c_{ii} = 0$ iff $a_{ik} b_{ik} \neq 0$ for even number of cases. The proof of the theorem follows immediately by noting that $a_{ik} b_{ik} \neq 0$ is equivalent to say that the i^{th} and k^{th} vertices are adjacent in both the graphs.

Lemma: 2.9 The $(i,j)^{th}$ $(i \neq j)$ entry of the matrix product $A(G)\,A(H)$ is either 0 or 1 depending on whether the number of GH paths from v_i to v_j is even or odd respectively.

Lemma: 2.10 The product $A(G)\,A(\bar{G})$ (G is a Petersen graph and $\bar{G}$ is any K-regular sub graph of G) is graphical if and only if the following statements are true

i) For every i $(1 \leq i \leq n)$, there are even number of vertices v_k such that $v_i \sim_G v_k$ and $v_k \sim_{\bar{G}} v_i$,

ii) For each pair of vertices v_i and v_j $(i \neq j)$ the number of GH paths and HG paths from v_i tov_j have same parity.

Example: 2.11 Consider a Petersen graph G and a 2 regular sub graph of its complement is shown in figure 1.2

$$A(G)A(\bar{G}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

and the cocktail parity graph shown in figure 1.3 is the graph realizing $A(G)\,A(\bar{G})$ is graphical.

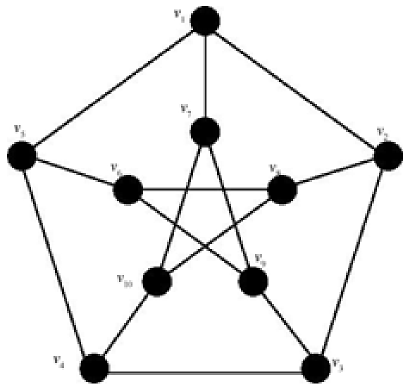

Figure-1.1

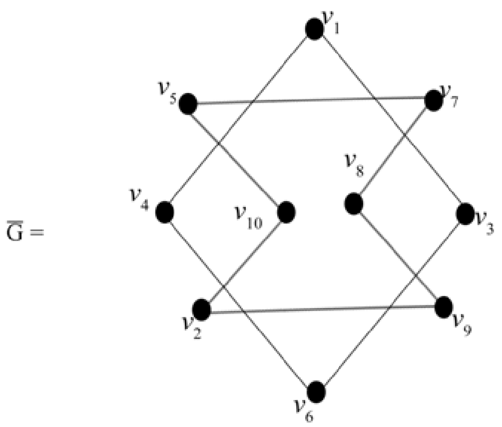

Figure-1.2

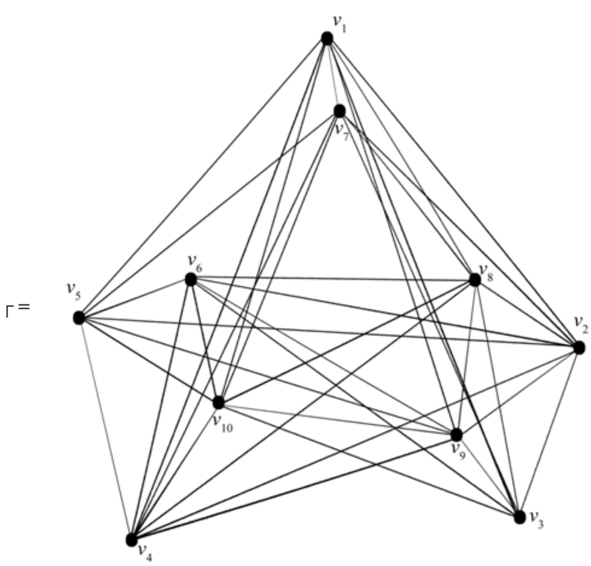

Figure-1.3

A H be any 2-regular sub graph of $\bar{G}$ on 10 vertices for which Γ is the graph realizing $A(G)A(\bar{G})$

Theorem: 2.12 For any graph G and its compliment $\bar{G}$ on the set of vertices $\{v_{1,}, v_2, \dots, v_n\}$ the following statements are equivalent

(i) The matrix product $A(G)$ $A(\bar{G})$is graphical.

(ii) For every i and j, $1 \leq i, j \leq n, deg_G v_i - deg_G v_j = 0$ (mod-2)

(iii) The graph G is parity regular.

Proof: Note that $(ii) <=> (iii)$ follows from the definition of parity regular graphs. Now, we shall prove$(i) <=> (ii)$.

Let $\left(A(G)\right)_{ij} = \left(a_{ij}\right)$ for all $i = 1,2, \dots, n \,; j = 1,2, \dots, n$ from the definitions of the complement of a graph and GH path,

$H = \bar{G}$ implies that

$deg_G v_i =$ Number of walks of length 2 from v_i to v_j in G + Number of $G\,\bar{G}$ paths from v_i to $v_j + a_{ij}$ (1)

Similarly, $deg_G v_j =$ Number of walks of length 2 from v_j to v_i in G + Number of $G\,\bar{G}$ paths from v_j to $v_i + a_{ji}$ (2) for every distinct pair of vertices v_i and v_j.

Since $G\,\bar{G}$ path from v_j to v_i is a $\bar{G}\,G$ path from v_i to v_j.

Comparing the right hand sides of (1) to (2), we get that $A(G)A(\bar{G})$ is graphical if and only if $deg_G v_i \equiv deg_G v_j$ $(mod$-2)

Remark: 2.13 It is also possible for one to prove $(i) => (ii)$, by taking $A(\bar{G}) = J - A(G) - I$ in the matrix products $A(G)A(\bar{G})$ and $A(\bar{G})\,A(G)$, where J is the $n \times n$ matrix with all 1's and I is the $n \times n$ identity matrix.

Theorem: 2.14 Consider a graph G and its complement $\bar{G}$ defined on the set of vertices $\{v_1, v_2, \dots v_n\}$. Then $A(G)A(\bar{G}) = 0$ and the diagonal value of $[A(G)]^2 = 1$ if $i = j$.

Theorem: 2.15 Let G be a graph and its complement $\bar{G}$ defined on the set of vertices $\{v_1, v_2, \dots v_n\}$.
Then $A(G)A(\bar{G}) = A(G)$ iff $[A(G)]^2$ is either a null matrix or the matrix J with all entries equal to 1.

Proof: Let, $A(G) = (a_{ij})$, we have

$deg_G v_i \equiv$ number of walks of length 2 from v_i to v_j in G(mod-2) for $i \neq j$ (A)

Now, G is a parity regular and therefore, $deg_G v_i - deg_G v_j \equiv 0\ (mod$-2)

So, from (A) we get that $\left(A(G)\right)^2$ is either 0 $or\ J$.

Conversely, Suppose that $\left(A(G)\right)^2$ is either $O\ or\ J$. If $\left(A(G)\right)^2 = 0$ we get that the degree of all the vertices in G are even and $\left(A(G)\right)^2 = J$ would mean that degree of all the vertices are odd. By taking $A(\bar{G}) = J - A(G) - I$
$= J + A(G) + I$ [since we know that the minus $(-)$ is the same as the plus (+) under modulo-2)]

Therefore, we get $A(G)A(\bar{G}) = A(G)(J + A(G) + I)$
$= A(G)J + \left(A(G)\right)^2 + A(G)$

In each case, $\left(A(G)\right)^2$ is $O\ or\ J$, we get that the right hand side of the above reduces to $A(G)$. Which also characterizes the graphs G with property $A(G)A(\bar{G}) = A(G)$ in terms of characteristics of $\bar{G}$.

Theorem: 2.16 Let G be a graph with adjacency matrix $A(G)$. Then the following statements are equivalent.

i) $\left(A(G)\right)^2 = A(G)$ $\quad i.e.,\ A(G)$ is idempotent
ii) $A(G)A(\bar{G}) = 0$ and the degree of every vertex in G is even.
iii) The number of $G\bar{G}$ paths of length 2 between every pair of vertices is even and the degree of every vertex in G is even.

Proof: $(i) => (ii)$. $\left(A(G)\right)^2$ is graphical implies that the diagonal entries of $\left(A(G)\right)^2$ are zeros, we get that degree of each vertex in G is zero (modulo-2).

In other words, degree of each $A(G)J = 0$.

Therefore, $A(G)A(\bar{G}) = A(G)(J - A(G) - I) = -\left(A(G)\right)^2 - A(G) = 0$

Whenever $\left(A(G)\right)^2 = A(G)$

$(ii) => (iii)$ follows from Theorem 2.15

$(iii) => (i)$ follows from Theorem 2.12

The graph G as shown in figure 1.1 such that $\left(A(G)\right)^2 = A(G)$ and the degree of every vertex of G is even. Further, $A(G)A(\bar{G}) = 0$.

REFERENCES

1. D.B.West Introduction to Graph theory Pearson Education (Singapore) Pte. Ltd. 2002.
2. F.Buckley and F.Harary, Distance in Graphs, Addison – Wesley publishing company, 1990.
3. Indian J. Pure Appl. Math. December 2014, 851-860. (C) Indian National Science Academy.
4. S.Akbari, F.Moazami and A.Mohammadian, Commutativity of the Adjacency Matrices of graphs, Discrete Mathematics, 309 (3) (2009), 595-600.
5. K.Manjunatha Prasad, G. Sudhakara, H.S. Sujatha and M. Vinay, Matrix Product of Graphs. In: R.B. Bapat et al., (Editors), Combinatorial Matrix Theory and Generalized Inverses of Matrices, Springer, 41-56, 2013.

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