

CONNECTED TOTAL DOMINATING SETS AND ITS POLYNOMIAL OF $K_n \times P_r$

B. STEPHEN JOHN*

Associate Professor, Department of Mathematics,
Annai Velankanni College, Tholayavattam, Tamil Nadu, India.

T. ANISHA ROSE (ST)

Department of Mathematics,
Annai Velankanni College, Tholayavattam, Tamil Nadu, India.

(Received On: 06-08-16; Revised & Accepted On: 30-08-16)

ABSTRACT

In this paper, we are going to study the connected total domination polynomial of $K_n \times P_r$. The connected total domination polynomial of a graph G of order n is defined $D_{ct}(G, x) = \sum_{i=\gamma_{ct}(G)} d_{ct}(G, i) x^i$, where $d_{ct}(G, i)$ is the number of connected total dominating sets of G with size i and $\gamma_{ct}(G)$ is the connected total domination number of G .

Keywords: Connected total dominating set, connected total domination number, connected total domination polynomial.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph of order n . For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V / uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \cup N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a connected total dominating set of G if every vertex $v \in V$ is adjacent to atleast one element of S and the induced sub graph $\langle S \rangle$ is connected. The connected total domination number $\gamma_{ct}(G)$ is called a γ -set.

The polynomial, $D_{ct}(G, x) = \sum_{i=\gamma_{ct}(G)} d_{ct}(G, i) x^i$ is defined as connected total domination polynomial of G .

where $d_{ct}(G, i)$ is the number of connected total dominating sets with size i .

2. CONNECTED TOTAL DOMINATION POLYNOMIALS

Definition: 2.1 A graph G consists of a pair $(V(G), E(G))$, where $V(G)$ is a non empty finite set whose elements are called points (or) vertices and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $E(G)$ are called lines or edges of the graph G .

Definition: 2.2 If $e = \{u, v\}$ is an edge of a graph G , written $e = uv$, we say that e joins the vertices u and v . Also we say that u and v are adjacent vertices, u and v are incident with e . If two vertices or not joined, then we say that they are not-adjacent.

Definition: 2.3 The graph G is complete if every two distinct vertices of G are adjacent. A complete graph with n vertices is denoted by K_n .

Definition: 2.4 A walk of a graph G is an alternating sequence of points and lines $v_0, x_1, v_1, x_2, v_2, \dots, v_{n-1}, x_n, v_n$ beginning and ending with points such that each line x_i is incident with v_{i-1} and v_i . A walk is called a path if all its points are distinct.

*Corresponding Author: B. Stephen John**
Associate Professor, Department of Mathematics,
Annai Velankanni College, Tholayavattam, Tamil Nadu, India.

Definition: 2.5 A subset S of vertices in a graph G is said to be a dominating set, if every vertex $v \in V - S$ is adjacent to atleast one element of S . A dominating set of G is said to be a total dominating set, if every vertex $v \in V$ is adjacent to atleast one element of S .

Definition: 2.6 A total dominating set of G is said to be a connected total dominating set, if the induced sub graph $\langle S \rangle$ of G is connected.

Theorem: 2.7 Let $G = K_n \times P_2$, then the total connected domination polynomial of G is,

$$D_{ct}(G, x) = nx^2[1 + x]^{2(n-1)} - n \left[2(n-1)C_1 - 2 \right] x^{2n-1} - (n-1)x^{2n} + 2x^n[1 + x]^n$$

Proof: $G_1 = K_n$ be the complete graph with n vertices, $G_2 = P_2$, its product $G = G_1 \times G_2$ is given in figure 1.1.

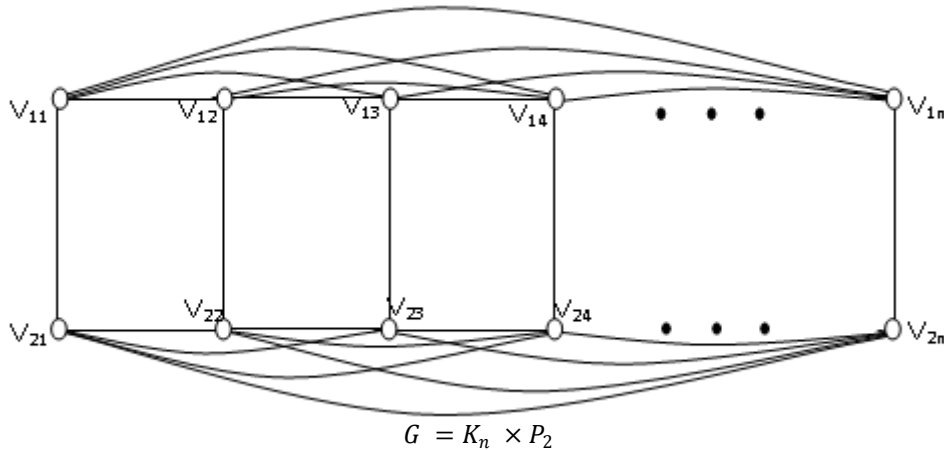


Figure 1.1

The vertices of G are denoted by $\{v_{11}, v_{12}, \dots, v_{1n}, v_{21}, v_{22}, \dots, v_{2n}\}$

Let, $S_i = \{V_{ki} / k = 1, 2; i = 1, 2, \dots, n\}$ and $T_j = \{V_{jk} / j = 1, 2; k = 1, 2, \dots, n\}$

The total connected dominating set with cardinality 2 are,
 $D_{ct}(G, 2) = \{S_i / i = 1, 2, \dots, n\}$

Therefore, $d_{ct}(G, 2) = n$

The total connected dominating set with cardinality 3 are,
 $D_{ct}(G, 3) = \{S_i \cup \{x_j\} / i, j = 1, 2, \dots, n; i \neq j\}$

Therefore, $d_{ct}(G, 3) = n \left[2(n-1)C_1 \right]$

The total connected dominating set with cardinality 4 are,
 $D_{ct}(G, 4) = \{S_i \cup \{x_j, x_k\} / i, j, k = 1, 2, \dots, n; i \neq j, k\}$

Therefore, $d_{ct}(G, 4) = n \left[2(n-1)C_2 \right]$

The total connected dominating set with cardinality 5 are,
 $D_{ct}(G, 5) = \{S_i \cup \{x_j, x_k, x_l\} / i, j, k, l = 1, 2, \dots, n; i \neq j, k, l\}$

Therefore, $d_{ct}(G, 5) = n \left[2(n-1)C_3 \right]$

Proceeding in this way, we get

The total connected dominating set with cardinality n are,
 $D_{ct}(G, n) = \{S_i \cup \{x_j, x_k, \dots, x_t\} / i, j, k, \dots, t = 1, 2, \dots, n; i \neq j, k, \dots, t\} \cup \{T_j / j = 1, 2\}$

Therefore, $d_{ct}(G, n) = n \left[\overline{2(n-1)C_{n-2}} \right] + 2$

The total connected dominating set with cardinality $n + 1$ are,

$$D_{ct}(G, n + 1) = \{S_i \cup \{x_j, x_k, \dots, x_t, x_{t+1}\} / i, j, k, \dots, t + 1 = 1, 2, \dots, n; \quad i \neq j, k, \dots, t + 1\} \\ \cup \{T_j \cup \{x_k\} / \{x_k\} \notin T_j, \quad j = 1, 2; \quad k = 1, 2, \dots, n; \quad j \neq k\}$$

Therefore, $d_{ct}(G, n + 1) = n \left[\overline{2(n-1)C_{n-1}} \right] + 2(nC_1)$

The total connected dominating set with cardinality $n + 2$ are,

$$D_{ct}(G, n + 2) = \{S_i \cup \{x_j, x_k, \dots, x_{t+2}\} / i, j, k, \dots, t + 2 = 1, 2, \dots, n; \quad i \neq j, k, \dots, t + 2\} \\ \cup \{T_j \cup \{x_k, x_l\} / \{x_k, x_l\} \notin T_j, \quad j = 1, 2; \quad k, l = 1, 2, \dots, n; \quad j \neq k, l\}$$

Therefore, $d_{ct}(G, n + 2) = n \left[\overline{2(n-1)C_n} \right] + 2(nC_2)$

Proceeding in this way, we get

The total connected dominating set with cardinality $2n - 2$ are,

$$D_{ct}(G, 2n - 2) = \{S_i \cup \{x_j, x_k, \dots, x_t, \dots, x_\alpha\} / i, j, \dots, \alpha = 1, 2, \dots, n; \quad i \neq j, \dots, \alpha\} \\ \cup \{T_j \cup \{x_k, x_l, \dots, x_\alpha\} / \{x_k, x_l, \dots, x_\alpha\} \notin T_j, \quad j = 1, 2; \quad k, l, \dots, \alpha = 1, 2, \dots, n; \quad j \neq k, l, \dots, \alpha\}$$

Therefore, $d_{ct}(G, 2n - 2) = n \left[\overline{2(n-1)C_{2(n-1)-2}} \right] + 2(nC_{n-2})$

The total connected dominating set with cardinality $2n - 1$ are,

$$D_{ct}(G, 2n - 1) = \{S_i - \{x_i\} / \{x_i\} \in S_i; \quad i = 1, 2, \dots, n\}$$

Therefore, $d_{ct}(G, 2n - 1) = 2n$

The total connected dominating set with cardinality $2n$ are,

$$D_{ct}(G, 2n) = 1$$

Therefore, $d_{ct}(G, 2n) = 1$

Hence, the total connected domination polynomial of G is,

$$D_{ct}(G, x) = nx^2 + n \left[\overline{2(n-1)C_1} \right] x^3 + n \left[\overline{2(n-1)C_2} \right] x^4$$

$$\dots + \left[n \left[\overline{2(n-1)C_{(n-2)}} \right] + 2 \right] x^n + \left[n \left[\overline{2(n-1)C_{(n-1)}} \right] + 2(nC_1) \right] x^{n+1} \\ \dots + \left[n \left[\overline{2(n-1)C_{2(n-1)-2}} \right] + 2(nC_{n-2}) \right] x^{2n-2} + 2nx^{2n-1} + x^{2n}$$

$$\Rightarrow D_{ct}(G, x) = \left[\begin{array}{l} nx^2 + n \left[\overline{2(n-1)C_1} \right] x^3 + n \left[\overline{2(n-1)C_2} \right] x^4 + \dots + n \left[\overline{2(n-1)C_{(n-2)}} \right] x^n \\ + n \left[\overline{2(n-1)C_{(n-1)}} \right] x^{n+1} + \dots + n \left[\overline{2(n-1)C_{2(n-1)-2}} \right] x^{2n-2} + 2nx^{2n-1} + x^{2n} \\ + [2x^n + 2(nC_1)x^{n+1} + \dots + 2(nC_{n-2})x^{2n-2}] \end{array} \right]$$

$$\Rightarrow D_{ct}(G, x) = \left[\begin{array}{l} nx^2 + n \left[\overline{2(n-1)C_1} \right] x^3 + n \left[\overline{2(n-1)C_2} \right] x^4 + \dots + n \left[\overline{2(n-1)C_{(n-2)}} \right] x^n \\ + n \left[\overline{2(n-1)C_{(n-1)}} \right] x^{n+1} + \dots + n \left[\overline{2(n-1)C_{2(n-1)-2}} \right] x^{2n-2} + n \left[\overline{2(n-1)C_{2(n-1)-1}} \right] x^{2n-1} \\ + n \left[\overline{2(n-1)C_{2(n-1)}} \right] x^{2n} - n \left[\overline{2(n-1)C_1} - 2 \right] x^{2n-1} - (n-1)x^{2n} \\ + [2x^n + 2(nC_1)x^{n+1} + \dots + 2(nC_{n-2})x^{2n-2}] \end{array} \right]$$

$$\Rightarrow D_{ct}(G, x) = nx^2[1 + x]^{2(n-1)} - n \left[\overline{2(n-1)C_1} - 2 \right] x^{2n-1} - (n-1)x^{2n} + 2x^n[1 + x]^n$$

Theorem: 2.8 Let $G = K_n \times P_r$, then the total connected domination polynomial of G is,

$$D_{ct}(G, x) = nx^r [1 + x]^{r(n-1)} - n[r(n-1)C_1 - r] x^{rn-1} - (n-1)x^{rn} + (r-2)[n(n-1)]x^{r+1}[1+x]^{(n-2)r} + x^{(r-2)n}[1+x]^{2n} \text{ for some } r > 2.$$

Proof: K_n be the complete graph with n vertices, P_r is a path of length r . Then its product $G = K_n \times P_r$ is given in figure 1.2

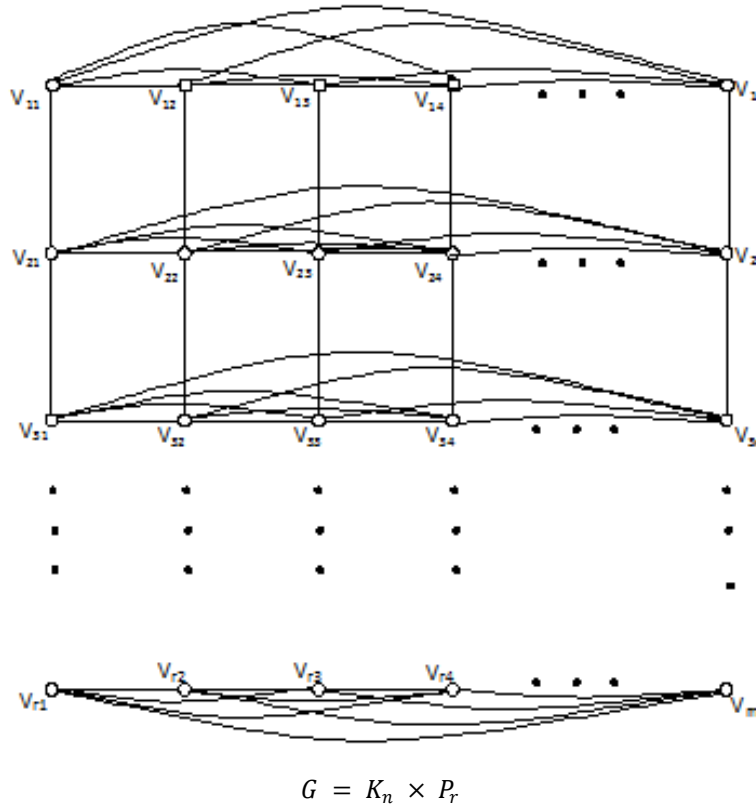


Figure: 1.2

The vertices of G is denoted by $\{V_{ij} / i = 1, 2, \dots, r; j = 1, 2, \dots, n\}$

Let, $S_i = \{V_{ki} / k = 1, 2, \dots, r; i = 1, 2, \dots, n\}$ and $T_j = \{V_{jk} / j = 1, 2, \dots, r; k = 1, 2, \dots, n\}$

The total connected dominating set with cardinality r are,
 $D_{ct}(G, r) = \{S_i / i = 1, 2, \dots, n\}$

Therefore, $d_{ct}(G, r) = n$

The total connected dominating set with cardinality $r + 1$ are,
 $D_{ct}(G, r + 1) = \{S_i \cup \{x_j\} / i, j = 1, 2, \dots, n; i \neq j\} \cup \{S_i - \{v_{ki}\} \cup \{v_{kj}\} / k = 1, 2, \dots, r; i, j = 1, 2, \dots, n; i \neq j\}$

Therefore, $d_{ct}(G, r + 1) = n[r(n-1)C_1] + (r-2)[n(n-1)]$

The total connected dominating set with cardinality $r + 2$ are,
 $D_{ct}(G, r + 2) = \{S_i \cup \{x_j, x_k\} / i, j, k = 1, 2, \dots, n; i \neq j, k\} \cup \{S_i \cup S_j - \{v_{ki}, v_{kj}\} \cup \{x_i\} / k = 1, 2, \dots, r; i, j = 1, 2, \dots, n; i \neq j\}$

Therefore, $d_{ct}(G, r + 2) = n[r(n-1)C_2] + (r-2)[n(n-1)][(n-2)rC_1]$

Proceeding in this way, we get

The total connected dominating set with cardinality $(r - 2)n$ are,

$$D_{ct}(G, (r - 2)n) = \{S_i \cup \{x_j, x_k, \dots, x_t\} / i, j, k, \dots, t = 1, 2, \dots, n; \quad i \neq j, k, \dots, t\} \\ \cup \{S_i \cup S_j - \{v_{ki}, v_{kj}\} \cup \{x_i, \dots, x_t\} / k = 1, 2, \dots, r; \quad i, j, \dots, t = 1, 2, \dots, n; \quad i \neq j, \dots, t\} \\ \cup \{T_j\} / j = 1, 2, \dots, r\}$$

Therefore, $d_{ct}(G, (r - 2)n) = n \overline{[r(n-1)C_{(r-2)(n-r)}]} + (r - 2)[n(n - 1)] \overline{[(n - 2)rC_{(r-2)(n-r-1)}]} + 1$

The total connected dominating set with cardinality $\overline{(r - 2)n + 1}$ are,

$$D_{ct}(G, \overline{(r - 2)n + 1}) = \{S_i \cup \{x_j, x_k, \dots, x_t, x_{t+1}\} / i, j, k, \dots, t + 1 = 1, 2, \dots, n; \quad i \neq j, k, \dots, t + 1\} \\ \cup \{S_i \cup S_j - \{v_{ki}, v_{kj}\} \cup \{x_i, \dots, x_{t+1}\} / k = 1, 2, \dots, r; \quad i, j, \dots, t + 1 = 1, 2, \dots, n; \quad i \neq j, \dots, t + 1\} \\ \cup \{T_j \cup \{x_k\} / \{x_k\} \notin T_j; \quad j = 1, 2, \dots, r; \quad k = 1, 2, \dots, n; \quad j \neq k\}$$

Therefore,

$$d_{ct}(G, \overline{(r - 2)n + 1}) = n \overline{[r(n-1)C_{(r-2)(n+1-r)}]} \\ + (r - 2)[n(n - 1)] \overline{(n - 2)rC_{(r-2)(n-r)}} + (2n)C_1$$

The total connected dominating set with cardinality $\overline{(r - 2)n + 2}$ are,

$$D_{ct}(G, \overline{(r - 2)n + 2}) = \{S_i \cup \{x_j, \dots, x_{t+2}\} / i, j, k, \dots, t + 2 = 1, 2, \dots, n; \quad i \neq j, k, \dots, t + 2\} \cup \\ \{S_i \cup S_j - \{v_{ki}, v_{kj}\} \cup \{x_i, \dots, x_{t+2}\} / k = 1, 2, \dots, r; \quad i, j, \dots, t + 2 = 1, 2, \dots, n; \quad i \neq j, \dots, t + 2\} \\ \cup \{T_j \cup \{x_k, x_l\} / \{x_k, x_l\} \in T_j; \quad j = 1, 2, \dots, r; \quad k, l = 1, 2, \dots, n; \quad j \neq k, l\}$$

Therefore, $d_{ct}G, \overline{(r - 2)n + 2}) = n \overline{[r(n-1)C_{(r-2)(n+2-r)}]} \\ + (r - 2)[n(n - 1)] \overline{(n - 2)rC_{(r-2)(n-r+1)}} + (2n)C_2$

Proceeding in this way, we get

The total connected dominating set with cardinality $rn - 2$ are,

$$D_{ct}(G, rn - 2) = \{S_i \cup \{x_j, x_k, \dots, x_t, \dots, x_\alpha\} / i, j, k, \dots, \alpha = 1, 2, \dots, n; \quad i \neq j, k, \dots, \alpha\} \cup \\ \{S_i \cup S_j - \{v_{ki}, v_{kj}\} \cup \{x_i, x_j, \dots, x_\alpha\} / k = 1, 2, \dots, r; \quad i, j, \dots, \alpha = 1, 2, \dots, n; \quad i \neq j, \dots, \alpha\} \cup \\ \{T_j \cup \{x_k, x_l, \dots, x_\alpha\} / \{x_k, x_l, \dots, x_\alpha\} \notin T_j; \quad j = 1, 2, \dots, r; \quad k, l = 1, 2, \dots, n; \quad j \neq k, l, \dots, \alpha\}$$

Therefore,

$$d_{ct}G, (rn - 2) = n \overline{[r(n-1)C_{r(n-1)-2}]} + (r - 2)[n(n - 1)] \overline{(n - 2)rC_{(n-2)-(n-2)}} + (2n)C_{2n-2}$$

The total connected dominating set with cardinality $rn - 1$ are,

$$D_{ct}G, (rn - 1) = \{S_i - \{x_i\} / \{x_i\} \in S_i, i = 1, 2, \dots, n\}$$

Therefore, $d_{ct}(G, rn - 1) = rn$

The total connected dominating set with cardinality rn are,

$$D_{ct}(G, rn) = 1$$

Therefore, $d_{ct}(G, rn) = 1$

Hence the total connected domination polynomial of G is

$$D_{ct}(G, x) = nx^r + \left[n \overline{[r(n-1)C_1]} + (r - 2)[n(n - 1)] \right] x^{r+1} \\ + \left[n \overline{[r(n-1)C_2]} + (r - 2)[n(n - 1)] \overline{[(n - 2)rC_1]} \right] x^{r+2} + \\ \dots + \left[n \overline{[r(n-1)C_{(r-2)(n-r)}]} + (r - 2)[n(n - 1)] \overline{[(n - 2)rC_{(r-2)(n-r-1)}]} + 1 \right] x^{(r-2)n} \\ + \left[n \overline{[r(n-1)C_{(r-2)(n+1-r)}]} + (r - 2)[n(n - 1)] \overline{[(n - 2)rC_{(r-2)(n-r)}]} + 2nC_1 \right] x^{\overline{(r+2)n+1}} \\ + \dots + \left[n \overline{[r(n-1)C_{r(n-1)-2}]} + (r - 2)[n(n - 1)] \overline{[(n - 2)rC_{(n-2)-(n-2)}]} + 2nC_{2n-2} \right] x^{rn-2} + mx^{rn-1} + x^{rn}$$

$$\Rightarrow D_{ct}(G, x) = [nx^r + n\overline{[r(n-1)C_1]}x^{r+1} + n\overline{[r(n-1)C_2]}x^{r+2} + \dots + n\overline{[r(n-1)C_{(r-2)(n-r)}}]x^{(r-2)n} \\ + n\overline{[r(n-1)C_{(r-2)(n+1-r)}}]x^{(r-2)n+1} + \dots + n\overline{[r(n-1)C_{r(n-1)-2}]x^{m-2} + mx^{m-1} + x^m] \\ + [(r-2)[n(n-1)]x^{r+1} + (r-2)[n(n-1)]\overline{[(n-2)rC_1]}x^{r+2} \\ + \dots + (r-2)[n(n-1)]\overline{[(n-2)rC_{(r-2)(n-r-1)}}]x^{(r-2)n} \\ + (r-2)[n(n-1)]\overline{[(n-2)rC_{(r-2)(n-r)}}]x^{(r-2)n+1} \\ + \dots + (r-2)[n(n-1)]\overline{[(n-2)rC_{(n-2)-(n-2)}}]x^{m-(r-1)}] \\ + [x^{(r-2)n} + 2nC_1x^{(r-2)n+1} + \dots + 2nC_{2n-2}x^{m-2}]$$

$$\Rightarrow D_{ct}(G, x) = [nx^r + n\overline{[r(n-1)C_1]}x^{r+1} + n\overline{[r(n-1)C_2]}x^{r+2} + \dots + n\overline{[r(n-1)C_{(r-2)(n-r)}}]x^{(r-2)n} \\ + n\overline{[r(n-1)C_{(r-2)(n+1-r)}}]x^{(r-2)n+1} + \dots + n\overline{[r(n-1)C_{r(n-1)-2}]x^{m-2} + n\overline{[r(n-1)C_{r(n-1)-1}]x^{m-1} \\ + n\overline{[r(n-1)C_{r(n-1)}}]x^m - n\overline{[r(n-1)C_1 - r]}x^{m-1} - (n-1)x^m] \\ + [(r-2)[n(n-1)]x^{r+1} + (r-2)[n(n-1)]\overline{[(n-2)rC_1]}x^{r+2} \\ + \dots + (r-2)[n(n-1)]\overline{[(n-2)rC_{(r-2)(n-r-1)}}]x^{(r-2)n} \\ + (r-2)[n(n-1)]\overline{[(n-2)rC_{(r-2)(n-r)}}]x^{(r-2)n+1} \\ + \dots + (r-2)[n(n-1)]\overline{[(n-2)rC_{(n-2)-(n-2)}}]x^{m-(r-1)}] \\ + [x^{(r-2)n} + 2nC_1x^{(r-2)n+1} + \dots + 2nC_{2n-2}x^{m-2}]$$

$$D_{ct}(G, x) = nx^r [1+x]^{r(n-1)} - n\overline{[r(n-1)C_1 - r]}x^{m-1} - (n-1)x^m + (r-2)[n(n-1)]x^{r+1}[1+x]^{(n-2)r} + x^{(r-2)n} [1+x]^{2n} \text{ for some } r > 2.$$

Hence the Proof.

The following Table represents the coefficients of the total domination polynomial of $G = K_n \times P_2$ for all $n < 8$.

$d_{ct}(G, i)$ G	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}
$K_3 \times P_2$	3	14	24	6	1										
$K_4 \times P_2$	4	24	62	88	72	8	1								
$K_5 \times P_2$	5	40	140	282	360	300	160	10	1						
$K_6 \times P_2$	6	60	270	720	1262	1524	1290	760	300	12	1				
$K_7 \times P_2$	7	84	462	1540	3465	5546	6482	5586	3535	1610	504	14	1		
$K_8 \times P_2$	8	112	728	2912	8008	16016	24026	27472	24080	16128	8148	3024	784	16	1

Table: 2.1

REFERENCES

1. Alikhani.S and Peng Y.H, Introduction to Domination Polynomial of a graph, arXiv: 0905-225[v] [math.co], 14 May (2009).
2. Alikhani.S, on the Domination Polynomial of some graph operations, ISRN combin, (2013).
3. Haynes .T.W, Hedetniemi. S.T and slater .P.J, Fundamentals of Domination in Graphs, Marcel Dekker, New York, (1998).
4. Saeid Alikhani and Emeric Deutsch, More on domination polynomial and domination root, arXiv:1305.3734v2, (2014).
5. Sahib sh.kabat, Abdul Jalil M. Khalaf and Roslan Hasni, Dominating sets and Domination Polynomials of Wheels, Australian Journal of Applied Science, (2014).
6. Vijayan .A and Anitha Baby .T, Connected Total domination polynomials of graphs, International journal of mathematical Archieve, 5(11), (2014).
7. Vijayan .A, Anitha Baby .T and Edwin .G, Connected Total dominating sets and connected total domination polynomials of stars and wheels, IOSR Journal of Mathematics (2014).

8. Vijayan .A, Anitha Baby .T, Connected Total dominating sets and connected total domination polynomials of Gem graphs, IJSIMR journal of mathematics (2015).

Source of support: Nil, Conflict of interest: None Declared

[Copy right © 2016. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]