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PRIME ACCESSIBLE RINGS

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ABSTRACT

In this paper, we prove that a 2- and 3- divisible prime accessible ring is either associative or commutative and a 2- and 3- divisible semiprime accessible ring is associative.

Keywords: *Prime ring, semiprime ring, accessible ring, n-divisible ring, nucleus.*

1. INTRODUCTION

Kleinfeld [1] studied the structure of standard and accessible rings. He proved that simple accessible rings are either associative or commutative. Thedy [3] studied the rings, in which the associator commutes with all elements of the ring i.e., ((w, x, y), z) = 0. He proved that simple nonassociative rings satisfying this identity are either associative or commutative.

In this paper we see that accessible rings satisfy the identity ((w, x, y), z) = 0. We prove that the associator and multiple of the associator are in the nucleus of an accessible ring. Using these properties, we show that a 2- and 3- divisible prime accessible ring is either associative or commutative and a 2- and 3- divisible semiprime accessible ring is associative.

2. PRELIMINARIES

A ring is defined to be accessible if the following two identities hold:

$$(x, y, z) + (z, x, y) - (x, z, y) = 0. \quad (1)$$

$$((w, x), y, z) = 0. \quad (2)$$

for all x, y, z in R, where the associator is defined as (x, y, z) = (xy)z – x(yz) for all x, y, z in R, the commutator is defined as (x, y) = xy – yx for all x, y in R.

Throughout this paper R represents an accessible ring. R is said to be prime whenever A and B are ideals of R such that AB=0, then either A=0 or B=0. R is said to be semiprime if for any ideal A of R, $A^2 = 0$ implies A=0. R is said to be n-divisible if nx = 0 imples x = 0 for all x in R and n, a natural number The nucleus N of R is defined as the set of all elements n in R such that (n, R, R) = (R, n, R) = (R, R, n) = 0.

By substituting z = y in (1), we obtain the flexible law

$$(y, x, y) = 0. \quad (3)$$

A linearization of the above identity yeilds

$$(y, x, z) = - (z, x, y). \quad (4)$$

Then $$(x, y, z) + (y, z, x) + (z, x, y) = 0. \quad (5)$$

In any arbitrary ring the identity

(xy, z) = x(y, z) + (x, z)y + (x, y, z) + (z, x, y) – (x, z, y) holds.

From (1) this identity becomes (xy, z) = x(y, z) + (x, z)y. (6)

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Another identity which holds in an arbitrary ring is

$$(wx, y, z) – (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z. \quad (7)$$

If n is an element of the nucleus N of R, then because of the flexible law (R, R, n) = 0. Finally because of (1) it follows that (R, n, R) = 0.

If n is substituted for w in (7), we obtain

$$(nx, y, z) = n(x, y, z), n\in N.$$

Combining this with (2) yeilds

$$(nx, y, z) = n(x, y, z) = (xn, y, z), n\in N. \quad (8)$$

We now proceed to develop further identities that hold in arbitrary accessible rings. The elements u, v, w, x, y, z will denote arbitrary elements of such rings.

By repeated use of (6), we break up ((w, x, y), z) as

$$\begin{aligned}((w, x, y), z) &= (wx\cdot y - w\cdot xy, z)\\ &= wx\cdot(y, z) + w(x, z)\cdot y + (w, z)x\cdot y - w\cdot x(y, z) - w\cdot(x, z)y – (w, z)\cdot xy\\ &= (w, x, (y, z)) + (w, (x, z),y) + ((w, z), x, y).\end{aligned}$$

Since (2) implies that every commutator is in the nucleus,

We obtain ((w, x, y), z) = 0. (9)

Hence every associator commutes with every element of R.

The associator ideal A of R is defined as A=Σ(R, R, R) + (R, R, R)R

Let S(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y).

In every ring we have the identities

$$(xy, z) + (yz, x) + (zx, y) = S(x, y, z), \quad (10)$$
$$((x, y), z) + ((y, z), x) + ((z, x), y) = S(x, y, z) – S(x, z, y). \quad (11)$$

3. MAIN RESULTS

First we prove some properties of the nucleus in R.

Lemma 1: If R is a prime accessible ring, then the associator is in the nucleus N of R.

Proof: From (8) and the fact that every commutator is in the nucleus, we get (v, x) (x, y, z) = ((v, x)x, y, z).

It follows from (6) that (vx, x) = v(x, x) + (v, x)x (or) (vx, x) = (v, x)x.

Consequently ((v, x)x, y, z) = ((vx, x),y, z) = 0.

Therefore (v, x) (x, y, z) = 0. (12)

A linearization of this identity becomes

$$(v, w) (x, y, z) = - (v, x) (w, y, z). \quad (13)$$

Using (6), (9), (13) and (2), we obtain

$$\begin{aligned}((v, w) (x, y, z), u) &= (v, w) ((x, y, z), u) + ((v, w), u) (x, y, z)\\ &= ((v, w), u) (x, y, z)\\ &= - (u, (v, w)) (x, y, z)\\ &= (u, x) ((v, w), y, z)\\ &= 0.\end{aligned}$$

Therefore ((v, w) (x, y, z), u) = 0.

Now using (8), (13) and (9), we obtain

$$\begin{aligned}((v, w) (x, y, z), t, u) &= (v, w) ((x, y, z), t, u)\\ &= - (v, (x, y, z)) (w, t, u)\\ &= 0.\end{aligned}$$

Thus $((v, w)(x, y, z), t, u) = 0$.

Since $(v, w) \in N$ we have $(v, w)((x, y, z), t, u) = 0$. (14)

Since R is prime and not commutative, (14) implies that $((x, y, z), t, u) = 0$.

By using the linearization of flexible property (3), we obtain $(u, t, (x, y, z)) = 0$. Finally because of (1), it follows that $(t, (x, y, z), u) = 0$. Therefore the associator (x, y, z) is in the nucleus N.

This completes the proof of the lemma.

Lemma 2: In an accessible ring R, $(R(R, R, R), R) = 0$.

Proof: By commuting (7) with r, we get

$$((wx, y, z), r) - ((w, xy, z), r) + ((w, x, yz), r) = (w(x, y, z), r) + ((w, x, y)z, r)$$

By using (9), we obtain

$$(w(x, y, z), r) + ((w, x, y)z, r) = 0.$$
$$(w(x, y, z), r) = -((w, x, y)z, r). \quad (15)$$

Equation (15) with w=y and using (3) gives

$$(y(x, y, z), r) = -((y, x, y)z, r) = 0.$$

So that $(y(x, y, z), r) = 0$. (16)

Linearization of (16) with y=w+y yeilds

$$(w(x, y, z), r) = -(y(x, w, z), r). \quad (17)$$

By substituting z=y in (15) and using (17) repeatedly, we get

$$\begin{aligned}(w(x, y, y), r) &= -((w, x, y)y, r)\\ &= ((w, y, y)x, r)\\ &= -((x, w, y)y, r)\\ &= ((y, w, x)y, r)\\ &= -((x, y, w)y, r)\\ &= ((y, x, y)w, r)\\ &= 0.\end{aligned}$$

i.e. $(w(x, y, y), r) = 0$. (18)

Linearization of (18) with y=y+z yeilds

$$(w(x, y, z), r) = -(w(x, z, y), r).$$

Using the linearization of flexible identity, the above equation yeilds

$$(w(x, y, z), r) = (w(y, z, x), r).$$

Similarly $(w(y, z, x), r) = (w(z, x, y), r)$.

Therefore $(w(x, y, z), r) = (w(y, z, x), r) = (w(z, x, y), r)$. (19)

Using (19) and (5) gives

$$0 = (w((x, y, z) + (y, z, x) + (z, x, y)), r)$$
$$0 = 3(w(x, y, z), r).$$

Since R is 3- divisible, we have $(w(x, y, z), r) = 0$.
i.e. $(R(R, R, R), R) = 0$.

This completes the proof of the lemma.

Theorem 1: A 2- and 3- divisible prime accessible ring is either associative or commutative.

Proof: By using (6) we get

$$(w(x, y, z), r) = w((x, y, z), r) + (w, r)(x, y, z).$$

By using (9) and lemma 2, we obtain

$$(w, r)\,(x, y, z) = 0. \tag{20}$$

We know that A is an ideal consisting of all finite sums of elements of the form (x, y, z) or of the form $w(x, y, z)$ and B is an ideal consisting of all finite sums of elements of the form (x, y) or of the form $w(x, y)$.

From (20), it follows that BA=0

Since R is prime, we have either B=0 or A=0.

If B=0, then R is commutative. If A=0, then R is associative.

Hence R is either commutative or associative.

Theorem 2: A 2- and 3- divisible semiprime accessible ring R is associative.

Proof: From lemma 1, we have

$$((x, y, z), r, s) = 0. \tag{21}$$

By taking associators of (7) and using (9) we get

$$(w(x, y, z), r, s) + ((w, x, y)z, r, s) = 0. \tag{22}$$

By substituting w=y in (22) and using (3)

$$(y(x, y, z), r, s) + ((y, x, y)z, r, s) = 0$$

So that $(y(x, y, z),r, s) = 0.$ (23)

By linearizing (23) with y=w+y yields

$$(w(x, y, z), r, s) + (y(x, w, z), r, s) = 0. \tag{24}$$

By substituting z = y in (22) and (24) we have

$$(w(x, y, y), r, s) + ((w, x, y)y, r, s) = 0. \tag{25}$$

and $(w(x, y, y), r, s) + (y(x, w, y), r, s) = 0.$ (26)

By subtracting the equation (26) from (25), we get

$$((w, x, y)y, r, s) – (y(x, w, y), r, s) = 0.$$

i.e. $((w, x, y)y, r, s) = (y(x, w, y), r, s).$ (27)

Since $(y(x, y, z), r, s) = 0$, we have $(y(y, x, z), r, s) = 0$ and $– (z(y, y, x), r, s) = 0.$

This implies that $(z(x, y, y), r, s) = 0.$

i.e. $((x, y, y)z, r, s) = 0.$ (28)

Linearization of (28) with y=w+y yeilds

$$((x, w, y)z, r, s) + ((x, y, w)z, r, s) = 0.$$

Using linearization of flexible identity, the above equation yeilds

$$((x, w, y)z, r, s) = ((w, y, x)z, r, s).$$

Similarly $((w, y, x)z, r, s) = ((y, x, w)z, r, s).$

Therefore $((x, w, y)z, r, s) = ((w, y, x)z, r, s) = ((y, x, w)z, r, s).$ (29)

Using (29) and (5) gives

$$0 = (((w, x, y) + (x, y, w) + (y, w, x))z, r, s)$$
$$0 = 3((w, x, y)z, r, s).$$

Since R is 3- divisible, we have $((w, x, y)z, r, s) = 0.$

Using (8) and lemma 1, we have $(w, x, y)\,(z, r, s) = 0.$ (30)

We know that A is an associator ideal consisting of all elements of the form (x, y, z) and w(x, y, z). From (30) it follows that $A^2=0$. Since R is semiprime, we obtain A=0. Hence R is associative.

This completes the proof of the theorem.

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