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INVERSE AND DISJOINT SECURE DOMINATING SETS IN GRAPHS

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ABSTRACT

Let D be a minimum secure dominating set of a graph G = (V, E). If V – D contains a secure dominating set D' of G, then D' is called an inverse secure dominating set with respect to D. The inverse secure domination number $\gamma_s^{-1}(G)$ of G is the minimum cardinality of an inverse secure dominating set of G. The disjoint secure domination number $\gamma_s\gamma_s(G)$ of a graph G is the minimum cardinality of the union of two disjoint secure dominating sets in G. In this paper, we establish some results for the inverse secure domination number. Also we initiate a study of the disjoint secure domination number and obtain some results on this new parameter.

Keywords: *Inverse domination number, inverse secure domination number, disjoint secure domination number.*

AMS Subject Classification: *05C69.*

1. INTRODUCTION

By a graph, we mean a finite, undirected, without loops, multiple edges and isolated vertices. Let $G = (V, E)$ be a graph with p vertices and q edges. For the general concepts, the reader may refer to [1]. A set D of vertices in a graph G is called a dominating set if every vertex in $V – D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Recently several domination parameters are given in the books by Kulli in [2,3,4]. Let D be a minimum dominating set of G. If $V – D$ contains a dominating set D' of G, then D' is called an inverse dominating set of G with respect to D. The inverse domination number $\gamma^{-1}(G)$ of G is the minimum cardinality of an inverse dominating set of G. The inverse domination in graphs was introduced by Kulli and Sigarkanti in [5]. Many other inverse domination parameters in domination theory were studied, for example, in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

A dominating set D in G is called a secure dominating set in G if for every vertex u in $V – D$, there exists v in D adjacent to u such that $(D - \{v\}) \cup \{u\}$ is a dominating set. The secure domination number $\gamma_s(G)$ of G is the minimum cardinality of a secure dominating set of G. This was introduced by Cockayne *et al.* in [20].

Let D be a minimum secure dominating set of G. If $V – D$ contains a secure dominating set D' of G, then D' is called an inverse secure dominating set with respect to D. The inverse secure domination number $\gamma_s^{-1}(G)$ is the minimum cardinality of an inverse secure dominating set of G. The inverse secure domination in graphs was found in the paper of Enriquez et al. in [21]. A γ_s^{-1}-set is a minimum inverse secure dominating set. Similarly other sets can be expected.

A dominating set D of G is a split dominating set if the induced subgraph $\langle D \rangle$ is disconnected. The split domination number $\gamma_{sd}(G)$ of G is the minimum cardinality of a split dominating set of G. This concept was introduced by Kulli and Janakiram in [22].

Let $\Delta(G)$ denote the maximum degree and $\lceil x \rceil$ the least integer greater than or equal to x. The complement of G is denoted by $\overline{G}$.

2. INVERSE SECURE DOMINATION

Proposition A [21]: Let G be a connected graph with $p \geq 4$ vertices. Then

$$\gamma_s(G) \leq \gamma_s^{-1}(G).$$

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Remark 2: Not all graphs have an inverse secure dominating set. For example, the path P_5 has a secure dominating set, but no inverse secure dominating set.

By Remark 2: Proposition A is not true for $p = 5$, Thus we have

Theorem 3: For any graph G with a γ_s^{-1} -set,

$$\gamma_s(G) \leq \gamma_s^{-1}(G) \tag{1}$$

and this bound is sharp.

Proof: Every inverse secure dominating set is a secure dominating set. Thus (1) holds.

The path P_4 achieves this bound.

Theorem 4: If a γ_s^{-1}-set exists in a graph G with p vertices, then

$$\gamma_s(G) + \gamma_s^{-1}(G) \leq p \tag{2}$$

and this bound is sharp.

Proof: This follows from the definition of $\gamma_s^{-1}(G)$.

The path P_4 achieves this bound.

Theorem 5: If a γ_s^{-1}-set exists in a graph G with p vertices, then

$$\gamma(G) + \gamma_s^{-1}(G) \leq p \tag{3}$$

and this bound is sharp.

Proof: By definition, $\gamma(G) \leq \gamma_s(G)$. By Theorem 4, $\gamma_s(G)+ \gamma_s^{-1}(G) \leq p$. Thus (3) holds.

The path P_4 and the cycle C_4 achieve this bound.

Theorem B [22]: For any graph G with an endvertex,

$$\gamma(G) = \gamma_{sd}(G).$$

We obtain a relation between $\gamma_{sd}(G)$ and $\gamma_s^{-1}(G)$.

Theorem 6: Let G be a graph with an endvertex. If a γ_s^{-1}-set exists in G with p vertices, then

$$\gamma_{sd}(G) + \gamma_s^{-1}(G) \leq p \tag{4}$$

and this bound is sharp.

Proof: From Theorem 5, we have $\gamma(G) + \gamma_s^{-1}(G) \leq p$. From Theorem B, we have $\gamma(G) = \gamma_{sd}(G)$. Thus (4) holds.

The path P_4 achieves this bound.

Theorem 7: For any graph G without isolated vertices and with an endvertex,

$$\gamma_{sd}(G) \leq \gamma_s(G) \tag{5}$$

and this bound is sharp.

Proof: From Theorem B, $\gamma(G) = \gamma_{sd}(G)$ and by definition $\gamma(G) \leq \gamma_s(G)$. Hence (5) holds.

The path P_4 achieves this bound.

Corollary 8: Let G be a graph with an endvertex. If a γ_s^{-1} -set exists in G, then

$$\gamma_{sd}(G) \leq \gamma_s^{-1}(G) \tag{6}$$

We obtain lower and upper bounds on $\gamma_s^{-1}(G)$.

Theorem 9: For any graph G with p vertices and with a γ_s^{-1}-set,

$$\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leq \gamma_s^{-1}(G) \leq \left\lceil \frac{p\Delta(G)}{\Delta(G)+1} \right\rceil. \tag{7}$$

Proof: It is known that $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leq \gamma(G)$ and since $\gamma(G) \leq \gamma_s^{-1}(G)$, we see that the lower bound in (7) holds.

By Theorem 4, we have

$$\gamma_s^{-1}(G) \le p - \gamma_s(G).$$

Since $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \le \gamma(G) \le \gamma_s(G)$ and the above inequality,

$$\gamma_s^{-1}(G) \le p - \left\lceil \frac{p}{\Delta(G)+1} \right\rceil.$$

Thus the upper bound in (7) holds.

Theorem C [21]: Let G be a connected graph with $p \ge 2$ vertices. Then $\gamma_s(G) = 1$ if and only if $G = K_p$.

We obtain the following bounds for $\gamma_s^{-1}(G)$.

Theorem 10: Let G be a connected graph with $p \ge 4$ vertices. If G has a γ_s^{-1}-set and $G \ne K_p$, then

$$2 \le \gamma_s^{-1}(G) \le p - 2 \qquad (8)$$

and these bounds are sharp.

Proof: Suppose G is connected and $G \ne K_p$. By Theorem C, $\gamma_s(G) \ge 2$. Since $2 \le \gamma_s(G)$ and by Theorem 3, $\gamma_s(G) \le \gamma_s^{-1}(G)$, we see that the lower bound of (8) follows.

By Theorem 4, we have $\gamma_s^{-1}(G) \le p - \gamma_s(G)$ and since $2 \le \gamma_s(G)$

$$\gamma_s^{-1}(G) \le p - 2.$$

Thus $\quad 2 \le \gamma_s^{-1}(G) < p - 2.$

The path P_4 achieves both lower and upper bounds.

Now we obtain a Nordhaus - Gaddum type result for secure domination number.

Theorem 11: Let G be a graph with $p \ge 4$ vertices and $G \ne K_p$. If a γ_s^{-1} -set exists and G and $\overline{G}$ have no isolated vertices, then

$$4 \le \gamma_s^{-1}(G) + \gamma_s^{-1}\left(\overline{G}\right) \le 2(p-2)$$

$$4 \le \gamma_s^{-1}(G)\gamma_s^{-1}\left(\overline{G}\right) \le (p-2)^2$$

and these bounds are sharp.

Proof: Since G and $\overline{G}$ have no isolated vertices and $G \ne K_p$,

$$2 \le \gamma_s^{-1}(G) \text{ and } 2 \le \gamma_s^{-1}\left(\overline{G}\right).$$

Thus both lower bounds follow.

By Theorem 10, we have

$$\gamma_s^{-1}(G) \le p-2 \text{ and } \gamma_s^{-1}\left(\overline{G}\right) \le p-2.$$

Thus both upper bounds follow.

The path P_4, $2K_2$ and cycle C_4 achieve these bounds.

3. DISJOINT SECURE DOMINATION

We introduce the concept of disjoint secure domination number.

Definition 12: The disjoint secure domination $\gamma_s\gamma_s(G)$ of a graph G is the minimum cardinality of the union of two disjoint secure dominating sets in G. We say that two disjoint secure dominating sets, whose union has cardinality $\gamma_s\gamma_s(G)$, is a $\gamma_s\gamma_s$-pair of G.

Remark 13: Not all graphs have a disjoint secure domination number. For Example, the cycle C_5 does not have two disjoint secure dominating sets.

Theorem 14: For any graph G with $\gamma_s^{-1}(G)$,

$$2\gamma_s(G) \leq \gamma_s\gamma_s(G) \leq \gamma_s(G) + \gamma_s^{-1}(G) \leq p.$$

Definition 15: A graph G is called $\gamma_s\gamma_s$-minimum if it has two disjoint γ_s-sets, that is, $\gamma_s\gamma_s(G) = 2\gamma_s(G)$.

Definition 16: A graph G is called $\gamma_s\gamma_s$-maximum if $\gamma_s\gamma_s(G) = p$.

The disjoint domination number $\gamma\gamma(G)$ of a graph G is the minimum cardinality of the union of two disjoint dominating sets in G, see [23]. Many other disjoint domination numbers were studied, for example, in [7, 8, 9, 14, 24].

When the disjoint secure domination number exists the following inequality holds.

Proposition 17: For any graph G with two disjoint secure dominating sets,

$$\gamma\gamma(G) \leq \gamma_s\gamma_s(G).$$

The cycle C_4, the paths P_2, P_4 achieve this bound.

The following results indicate the disjoint secure domination numbers of some standard graphs.

Proposition 18: For the complete graph K_p, $p\geq 2$,

$$\gamma\gamma(K_p) = \gamma_s\gamma_s(K_p) = 2\gamma_s(K_p) = 2.$$

Proposition 19: For the complete bipartite graph $K_{m,n}$, $4 \leq m \leq n$,

$$\gamma_s\gamma_s(K_{m,n}) = 2\gamma_s(K_{m,n}) = 8.$$

The complete graphs K_p, $p\geq 2$ and the complete bipartite graphs $K_{m,n}$, $4 \leq m \leq n$ are $\gamma_s\gamma_s$ -minimum.

The graphs K_2 and $K_{4,4}$ are $\gamma_s\gamma_s$ -maximum.

4. SOME OPEN PROBLEMS

Many questions are suggested by this research among them are the following:

Problem 1: Characterize graphs G for which $\gamma_s(G) = \gamma_s^{-1}(G)$.

Problem 2: Characterize graphs G for which $\gamma_s(G) + \gamma_s^{-1}(G) = p$.

Problem 3: Characterize graphs G for which $\gamma\gamma(G) = \gamma_s\gamma_s(G)$.

Problem 4: Characterize graphs G for which $\gamma_s\gamma_s(G) = 2\gamma(G)$.

Problem 5: Characterize the class of $\gamma_s\gamma_s$ -minimum graphs.

Problem 6: Characterize the class of $\gamma_s\gamma_s$ -maximum graphs.

Problem 7: Obtain bounds for $\gamma_s\gamma_s(G) + \gamma_s\gamma_s(\overline{G})$.

Problem 8: What is the complexity of the decision problem corresponding to $\gamma_s\gamma_s(G)$?

Problem 9: Is DISJOINT SECURE DOMINATION NP-complete for a class of graphs?

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