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SEMI - PRE R_1 AND WEAKLY SEMI-PRE R_0 SPACES

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ABSTRACT

In this paper we introduce sp-R_1 and weakly –sp- R_0 space with the help of semi-preopen sets defined by Andrijevic/[1]. Semi-pre θ-closure of a set is defined and used to investigate basic properties of sp-R_1 space. Some results on invariance and productivity of weakly - sp- R_0 spaces have been obtained.

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1. INTRODUCTION

In 1943, N. Shanin [10] introduced a new separation axiom termed as R_0 and in 1961, Davis [5] introduced the R_1-axiom. In 1963 Levine [11] defined semi-open set and Maheswari [7] *et al.* introduced $(R_0)_s$ space with the aid of semi-open sets while Caldas *et al.* [4] defined R_0 and R_1 spaces utilising preopen sets of Mashhour [9]. On the otherhand J.D.Maio [8] introduced weakly R_0 space and Arya *et al.* [2] defined weakly semi-R_0 using semi-open sets. Bandyopadhyay *et al.* [3] defined sp- R_0 space using semi- preopen sets introduced by Andrejevic/ [1]. This paper is the continuation of our study on separation axiom by introducing sp-R_1 space and weakly-sp- R_0 space using semi- preopen sets. In section 2 of this paper some known definitions and results are given which will be required in the sequel. Section 3 and section 4 deal with the definitions and characterisation along with some basic properties of sp- R_1 and weakly -sp- R_0 spaces respectively.

2. PRELIMINARIES

Throughout the paper (X, τ) or X always denotes a non trivial topological space. The family of all open sets containing x is denoted by $\Sigma(x)$. Interior and closure of a subset A of X is denoted by Int(A) and Cl(A) respectively.

Definition 2.1: $A \subset X$ is called a semi-preopen set (briefly s.p.o. set) [1] iff $A \subset Cl(Int(Cl(A)))$. The family of all s.p.o. sets is denoted by SPO(X). For each $x \in X$, the family of all s.p.o. sets containing x is denoted by SPO(X, x).

Definition 2.2: The complement of a s.p.o. set is called semi-preclosed [1].

Definition 2.3: The semi-preclosure [1] of $A \subset X$ is denoted by spcl (A) and is defined by spcl (A) = $\cap$ {B: B is semi-preclosed and $B \supset A$}.

Definition 2.4: A topological space X is said to be sp - T_1 [6] iff for every pair of points $x, y \in X$ such that $x \neq y$, there exist a $U \in SPO(X, x)$ not containing y and a $V \in SPO(X, y)$ not containing x.

Definition 2.5: A topological space X is said to be sp - T_2 [6] iff for every pair of distinct points $x, y \in X$ there exist disjoint sets $U \in SPO(X, x)$ and $V \in SPO(X, y)$.

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Definition 2.6: A space X is said to be a semi-pre R_0 [3] (briefly sp-R_0) space if spcl $(\{x\}) \subset U$ for every $U \in$ SPO (X, x).

Definition 2.7: Let $A \subset X$. Then the semi-pre Kernel [3] of A (briefly sp-Ker (A)) is defined to be the set sp-Ker (A) = $\cap$ {U: U $\in$ SPO (X), A $\subset$ U}.

Definition 2.8: A space X is called weakly-R_0 [8] iff $\cap$ {Cl ({x}) : $x \in X$} = ϕ.

Theorem 2.1 [3]: A topological space is sp-R_0 iff it is sp-T_1.

Theorem 2.2 [3]: A topological space X is sp-T_1 iff every one pointic set is semi-preclosed.

Lemma 2.1 [6]: If $A \in$ SPO (X) and $B \in$ SPO (Y) then, $A \times B \in$ SPO ($X \times Y$).

3. SEMI - PRE R_1 SPACES

We start with the definition of a sp-R_1 space, which runs as follows:

Definition 3.1: A topological space X is said to be semi-pre-R_1 (briefly sp-R_1) if for every pair of points x, y $\in$ X with spcl ({x}) $\neq$ spcl ({y}) there exist two disjoint sets U $\in$ SPO (X, x), V $\in$ SPO (X, y) such that spcl ({x}) $\subset$ U and spcl ({y}) $\subset$ V.

Theorem 3.1: Every sp-R_1 space is sp-R_0.

Proof: Let U $\in$ SPO (X, x) and y $\notin$ U. This gives x $\notin$ spcl ({y}) $\Rightarrow$ spcl ({x}) $\neq$ spcl ({y}).Since X is sp-R_1 there exists V $\in$ SPO (X, y) such that spcl ({y}) $\subset$ V and x $\notin$ V. Thus y $\notin$ spcl ({x}).The non-containment condition regarding y induces spcl ({x}) $\subset$ U. Hence X is sp-R_0.

Theorem 3.2: A topological space is sp-R_1 iff it is sp-T_2.

Proof: Let X be sp-R_1. Theorem 3.1 ensures that X is sp-R_0 and hence by Theorem 2.1, X is sp-T_1. We assert that X is sp-T_2. To this end let x, y $\in$ X with x $\neq$ y. Now sp-T_1-ness of X guarantees by Theorem 2.2 that spcl ({x}) = {x} and spcl ({y}) = {y}.Thus spcl ({x}) $\neq$ spcl ({y}). Therefore sp-R_1-ness of X provides two disjoint s.p.o. sets U and V such that x $\in$ U and y $\in$ V. Hence X is sp-T_2.

Definition 3.2: For A $\subset$ X, the semi pre θ-closure of A, denoted by $spcl_\theta$ (A), is defined by $spcl_\theta$ (A) ={x $\in$ X; spcl (V) $\cap$ A $\neq \phi$ for every $V \in$ SPO (X, x)}.

A is called semi-pre θ-closed if $spcl_\theta$ (A) = A.

Lemma 3.1: For any subset A of a topological space X, spcl (A) $\subset$ $spcl_\theta$ (A).

Proof is straight forward and is omitted.

Lemma 3.2: Let (X, τ) be a topological space and x, y $\in$ X. Then y $\in$ $spcl_\theta$ ({x}) iff x $\in$ $spcl_\theta$ ({y}).

Proof: Let y $\in$ $spcl_\theta$ ({x}). If possible suppose x $\notin$ $spcl_\theta$ ({y}).This guarantees the existence of a U $\in$ SPO (X, x) such that spcl (U) $\cap$ {y} = ϕ $\Rightarrow$ y $\notin$ spcl (U).Then there exists a V $\in$ SPO (X, y) such that V $\cap$ U = ϕ $\Rightarrow$ spcl (V) $\cap$ U = ϕ $\Rightarrow$ spcl (V) $\cap$ {x} = ϕ $\Rightarrow$ y $\notin$ $spcl_\theta$ ({x})$\Rightarrow$ a contradiction .Thus y $\in$ $spcl_\theta$ ({x}) $\Rightarrow$ x $\in$ $spcl_\theta$ ({y}).

The proof of the converse part follows by pursuing the same argument.

Theorem 3.3: A topological space X is sp-R_1 iff spcl ({x}) = $spcl_\theta$ ({x}) for every $x \in X$.

Proof: Assume X be sp-R_1. If possible suppose there exists a point $x \in X$ such that spcl({x}) $\neq$ $spcl_\theta$ ({x}). By Lemma 3.1 spcl ({x}) $\subset$ $spcl_\theta$ ({x}).This guarantees the existence of a y $\in$ X such that y $\in$ $spcl_\theta$ ({x}) but y $\notin$ spcl ({x}). Hence spcl ({x}) $\neq$ spcl ({y}). Again sp-R_1-ness of X provides us U_1 $\in$ SPO (X, x), U_2 $\in$ SPO (X, y) such that spcl ({x}) $\subset$ U_1, spcl ({y}) $\subset$ U_2 and $U_1 \cap U_2 = \phi$. Thus {x} $\cap$ spcl (U_2) = ϕ $\Rightarrow$ y $\notin$ $spcl_\theta$ ({x}) $\Rightarrow$ a contradiction.Therefore, the foregoing gives $spcl_\theta$ ({x}) = spcl ({x}).

Conversely, suppose that the given condition holds for every $x \in X$. We assert that X is sp-R_0. To this end let $x \in X$ and $U \in SPO(X, x)$. We take $y \notin U$. Obviously $spcl_\theta(\{y\}) = spcl(\{y\}) \subset X - U. \Rightarrow x \notin spcl_\theta(\{y\})$ So, Lemma 3.2 induces that $y \notin spcl_\theta(\{x\})$.Using Lemma3.1 one infers that $y \notin spcl(\{x\}) \Rightarrow spcl(\{x\}) \subset U$. Hence X is sp-$R_0$. Therefore by Theorem 2.1, X is sp-T_1. Next let $\alpha, \beta \in X$ with $\alpha \neq \beta$. By Theorem 2.2 $spcl(\{\alpha\}) = \{\alpha\}$ and $spcl(\{\beta\}) = \{\beta\}$.Clearly $\beta \notin spcl(\{\alpha\}) = spcl_\theta(\{\alpha\})$. Therefore there exists a $V \in SPO(X, \beta)$ such that $spcl(V) \cap \{\alpha\} = \phi \Rightarrow \alpha \in X - spcl(V) \in SPO(X)$.Thus for every $\alpha, \beta \in X$ with $\alpha \neq \beta$ there exist $X - spcl(V) \in SPO(X, \alpha)$, $V \in SPO(X, \beta)$such that $(X - spcl(V)) \cap V = \phi$.This indicates that X is sp-T_2 and hence, by Theorem 3.2, X is sp-R_1

4. WEAKLY SEMI-PRE R_0 SPACES

Definition 4.1: A topological space X is said to be weakly semi-pre-R_0 (briefly wsp-R_0) iff $\cap\{spcl(\{x\}) : x \in X\} = \phi$.

Remark 4.1: Obviously every sp-R_0 space is wsp-R_0 but the converse need not be true as the following shows.

Example 4.2: Let X = {a, b, c, d} be the set with the topology $\tau = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$.Then X is wsp-$R_0$ but not sp-R_0.

Remark 4.2: Every weakly R_0 space is wsp-R_0 follows from the fact that $\cap\{spcl(\{x\}): x \in X\} \subseteq \cap\{Cl(\{x\}): x \in X\}$. But the reverse relation does not hold in general which is clear from the following example.

Example 4.3: Let X = {a, b, c} be the set with the topology $\tau = \{\phi, X, \{a\}\}$.Then $SPO(X) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Here $\cap\{Cl(\{x\}): x \in X\} = \{b, c\} \neq \phi$ but $\cap\{spcl(\{x\}): x \in X\} = \phi$,which shows that (X, τ) is wsp-R_0 but not weakly R_0.

Remark 4.3: Maio [8] showed that a set equipped with the point exclusion topology cannot be weakly R_0. On the other hand, this space may be wsp-R_0 as shown below.

Example 4.4: Let X = {a, b, c} be the set with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is wsp-R_0 but not weakly R_0.

Theorem 4.1: A topological space X is wsp-R_0 iff sp-Ker $(\{x\}) \neq X$ for any $x \in X$.

Proof: Necessity: Suppose the theorem is false. Then there exists a $x_0 \in X$ such that sp-Ker $(\{x_0\}) = X$. This yields sp-Ker $(\{x_0\}) = \cap\{G: G \in SPO(X, x_0)\} = X$, which indicates that X is the only s.p.o. set containing x_0.This reveals that every semi-preclosed subset of X contains x_0. Thus $x_0 \in spcl(\{x\})$ for any $x \in X$. Therefore $\cap\{spcl(\{x\}): x \in X\} \neq \phi \Rightarrow$ a contradiction to the hypothesis that X is wsp-R_0. Hence sp-Ker $(\{x\}) \neq X$ for any $x \in X$.

Sufficiency: Suppose sp-Ker $(\{x\}) \neq X$ for every $x \in X$. If possible suppose X is not wsp-R_0 which means $\cap\{spcl(\{x\}): x \in X\} \neq \phi$.Then there exists a $x_0 \in X$ such that $x_0 \in \cap\{spcl(\{x\}): x \in X\}$.This implies that $x_0 \in spcl(\{x\})$ for every $x \in X$. Let $U \in SPO(X, x_0)$. Then from above $U \cap \{x\} \neq \phi$ for every $x \in X \Rightarrow x \in U$ for every $x \in X \Rightarrow X \subset U \Rightarrow U = X$.This then ensures sp-Ker $(\{x_0\}) = X \Rightarrow$ a contradiction to the assumption $\Rightarrow$ X is wsp-R_0.

We need the following definition and the lemma to establish the invariance of wsp-R_0-ness.

Definition 4.2: A mapping $f: X \to Y$ is called sp-closed iff $f[A] \in SPF(Y)$ for all $A \in SPF(X)$.

Lemma 4.1: If $f: X \to Y$ is a sp-closed function then $spcl_Y(\{f(x)\}) \subset f[spcl_Y(\{x\})]$ for every $x \in X$.

Proof: For any $x \in X, \{x\} \subset spcl_Y(\{x\})$ $f[\{x\}] \subset f[spcl_X(\{x\})]$.This gives $\{f(x)\} \subset f[spcl_X(\{x\})] \Rightarrow spcl_Y(\{f(x)\}) \subset spcl_Y(f[spcl_X(\{x\})]$.Since f is sp-closed $spcl_Y(f[spcl_X(\{x\})]) = f[spcl_X(\{x\})]$. From above $spcl_Y(\{f(x)\}) \subset f[spcl_X(\{x\})]$.

Theorem 4.2: If $f: X \to Y$ is an injective sp-closed mapping where X is wsp-R_0, then Y is so.

Proof: The injectivity of f yields $\cap\{spcl_Y(\{y\}): y \in Y\} \subset \cap\{spcl_Y(\{f(x)\}): x \in X\}$.The sp-closedness of f gives, by Lemma 4.1, $spcl_Y(\{f(x)\}) \subset f[spcl_X(\{x\})]$. So from above $\cap\{spcl_Y(\{y\}): y \in Y\} \subset \cap\{f[spcl_X(\{x\})]: x \in X\}$. Again the injectivity of f yields $\cap\{f[spcl_X(\{x\})]: x \in X\} \subset f[\cap\{spcl_X(\{x\}): x \in X\}]$. Now wsp-$R_0$-ness of X gives $\cap\{spcl_X(\{x\}): x \in X\} = \phi$. From the foregoing $\cap\{spcl_Y(\{y\}): y \in Y\} \subset f[\cap\{spcl_X(\{x\}): x \in X\}] = f[\phi] = \phi$. Hence Y is wsp-$R_0$.

PRODUCTIVITY OF wsp-R_0 SPACES

Lemma 4.2: Let $X = \Pi X_i$ be the product spaces of X_i's, i = 1, 2. …, n. Then for any point $<x_i> \in X$

$$spcl_X(\{<x_i>\}) \subset \Pi\ spcl_{X_i}(\{x_i\}),\ i=1,2,\dots,n.$$

Proof: Let $<\alpha_i> \in spcl_X(\{<x_i>\})$. Also let $U_i \in SPO(X_i, \alpha_i)$ and $U = \Pi\ U_i$. Lemma 2.1 gives $U \in SPO(X)$. Obviously $<\alpha_i> \in U$.

Now $<\alpha_i> \in spcl_X(\{<x_i>\}) \Rightarrow \{<x_i>\} \cap U \neq \phi$
$\Rightarrow \{x_i\} \cap U_i \neq \phi$, i = 1, 2, …, n $\Rightarrow \alpha_i \in spcl_{X_i}(\{x_i\})$, i = 1, 2, …, n $\Rightarrow <\alpha_i> \in \Pi\ spcl_{X_i}(\{x_i\})$ i=1,2,…,n.

So, $spcl_X(\{<x_i>\}) \subset \Pi\ spcl_{X_i}(\{x_i\})$ i=1,2,…,n.

Lemma 4.3: Let $X = \Pi\ X_i$ be the product space of X_i's, i = 1, 2, …, n. Then for any point $<x_i> \in X$

$$\bigcap_{<x_i> \in X} [\Pi\ spcl_{X_i}(\{x_i\})] = \Pi\ [\bigcap_{<x_i> \in X} spcl_{X_i}(\{x_i\})].$$

Proof: Let $<\alpha_i> \in \bigcap_{<x_i> \in X} [\Pi\ spcl_{X_i}(\{x_i\})]$

Then $<\alpha_i> \in \Pi\ spcl_{X_i}(\{x_i\})\ \forall <x_i> \in X$.

$\Rightarrow \alpha_i \in spcl_{X_i}(\{x_i\})\ \forall x_i \in X_i$, i = 1, 2, …, n.

$\Rightarrow \alpha_i \in \bigcap_{<x_i> \in X} spcl_{X_i}(\{x_i\})$, i=1,2,…,n.

$\Rightarrow <\alpha_i> \in \Pi\ [\bigcap_{<x_i> \in X} spcl_{X_i}(\{x_i\})]$

This gives

$$\bigcap_{<x_i> \in X} [\Pi\ spcl_{X_i}(\{x_i\})] \subset \Pi\ [\bigcap_{<x_i> \in X} spcl_{X_i}(\{x_i\})].$$

Theorem 4.3: A space $X = \Pi X_i$ (i=1, 2,…, n) is wsp-R_0, if one of the X_i is wsp-R_0.

Proof: Let X_K be wsp-R_0, for some fixed index K, where $1 \le K \le n$. The Lemma 4.2 yields

$$\bigcap_{<x_i> \in X} spcl(\{<x_i>\}) \subseteq \bigcap_{<x_i> \in X} [\Pi\ spcl_{X_i}(\{x_i\})].$$

An application of Lemma 4.3 gives

$$\bigcap_{<x_i> \in X} spcl_X(\{<x_i>\}) \subset \Pi\ [\bigcap_{<x_i> \in X} spcl(\{x_i\})].$$

Now wsp-R_0-ness of X_K ensures that

$\bigcap_{<x_i> \in X} spcl(\{x_K\}) = \phi$. Therefore, from above

$$\bigcap_{<x_i> \in X} spcl(\{<x_i>\}) \subset X_1 \times \dots \times X_{K-1} \times \phi \times X_{K+1} \times \dots \times X_n = \phi.$$

Hence X is wsp-R_0.

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