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COMMON FIXED POINTS OF MAPPINGS SATISFYING TRIANGLE INEQUALITY OF INTEGRAL TYPE

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ABSTRACT

The purpose of this paper is to consider a new approach for obtaining common fixed point theorems in metric spaces by subjecting the triangle inequality to a contractive condition of integral type. We use the concept of property (E.A) and R *-weak commutativity there, without the assumption of completeness of the space and the continuity of the mappings. Our results generalize and extend the results of Pant et.al [14] and others.*

Ams(Mos) Subject Classifications: *54H25, 47H10.*

Key Words: *Noncompatible maps, contractive condition, property (E.A). R-weak commutativity.*

1. INTRODUCTION

With the advent of the notion of compatible maps due to Jungck [5]. The study of common fixed point theorems for contractive type maps emerged as an area of vigorous research centered around the study of compatible maps and its weaker forms. However the study of common fixed points of noncompatible maps is equally interesting. Pant [9-13] initiated work along these lines by using the notion of pointwise R-weakly commuting mappings.

Definition 1.1[9]: Self-maps f and g of a metric space (X,d) are R-weakly commuting at a point $x \in X$ if $d(ASx, SAx) \le Rd(Ax, Sx)$ for some $R > 0$. They are pointwise R- weakly commuting on X if given $x \in X$ there exists $R > 0$ such that $d(ASx, SAx) \le Rd(Ax, Sx)$.

Describing the importance of pointwise R- weakly commuting maps in fixed point consideration. Pant [11] proved that the notion of pointwise R- weak commutativity is equivalent to commutativity at coincidence points. Junck [6] defines the commutativity at coincidence point with the notion of weakly compatible maps.

Definition 1.2 [6]: Self-maps f and g of a metric space (X,d) are weakly compatible if they commute at their coincidence point, that is $fgu = gfu$ whenever $fu = gu$ for $u \in x$.

Thus two maps are weakly compatible if and only if they are pointwise R- weakly commuting mappings. However pointwise R- weakly commuting maps need not be compatible as shown in Example 1 of [14].

Further generalizing the concept of noncompatible maps Amri and Moutawakil [2] have introduced a new property called property (E.A). So that compatible and noncompatible maps may be studied together.

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Definition 1.3 [2]: Let f and g be two self-maps of a metric space (X, d). Then they are said to satisfy the property (E.A), if there exists a sequence x_n in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \text{ for some } t \in X$$

If two maps are noncompatible then they satisfy the property (E.A). The converse however is not necessarily true.

Example 1.1: Let $X = [0, +\infty)$. Define $f, g : X \to X$ by $fx = \dfrac{x}{2}, \; gx = \dfrac{3x}{2}, \; \forall x \in X$. Consider the sequence $\{x_n\} = \frac{1}{n}, n \in N$. Clearly $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$. Thus f and g satisfy the property (E.A), and also they are weakly compatible since they commute at their coincidence point 0 and pointwise R-weakly commuting, but f and g are not necessarily noncompatible.

Example 1.2: Let $X = \mathbb{R}$. Define $f, g : X \to X$ by $fx = \dfrac{x}{3}, \; x \in X, \; gx = x^2, \; x \in X$. here 0 and $1/3$ are two coincidence points for the maps f and g. Note that f and g commute at 0 i.e. $fg(0) = gf(0) = 0$, but $fg(1/3) = 1/27$ and $gf(1/3) = 1/8$. So f and g are not weakly compatible. Consider the sequence $\{x_n\} = \left\{\dfrac{1}{n} : n \in N\right\}$ we have $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$. Thus f and g satisfy property (E.A).

Example 1.3: Let $X = [2, \infty)$. Define $f, g : X \to X$ by $fx = \dfrac{x}{2} + 1, \; gx = \dfrac{2x+2}{3}, \; \forall x \in [2, \infty)$. Here 2 is the coincidence point for the maps f and g. Note that f and g commute at 2 since $fg(2) = gf(2) = 2$. So f and g are weakly compatible. Suppose that property(E.A) holds. Then, there exists a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Therefore $\lim_{n\to\infty} x_n = 2t - 1$ and $\lim_{n\to\infty} x_n = \dfrac{3t-2}{2}$. Then $t = 0$ which is a contradiction since $0 \notin X$. Hence, f and g do not satisfy property (E.A).

Notice that from the Example 1.2-1.3 we can see that weakly compatible and property(E.A) are independent to each other.

In 2002 Branciari [3] obtained the fixed point theorem for a mapping satisfying an analogue of Banach contraction principle for integral type inequality stated as follows.

Define $\Phi = \{\phi : R^+ \to R^+$ is a lebesgue integrable mapping which is summable, non-negative$\}$ and φ satisfies the following inequality

$$\int_0^{\varepsilon} \varphi(\mathrm{t})\mathrm{dt} > 0 \quad \text{for each } \varepsilon > 0. \tag{1.1}$$

Theorem 1.1 [3]: Let (X, d) be a complete metric space, $c \in [0,1)$ and if $f : X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx, fy)} \varphi(\mathrm{t})\mathrm{dt} \le c \int_0^{d(x,y)} \varphi(t)dt,$$

where $\varphi \in \Phi$ and satisfy the condition (1.1) then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$

This result was further generalized by Rhoades [8], Aliouche [1], Phatak *et.al* [15], Gairola-Rawat [4] and others and also see the references thereof.

In the present paper using the notion of pointwise R - weak commutativity we demonstrate that triangle inequality can be used to establish common fixed point theroem by subjecting it to contractive condition of integral type. We use the concept of property(E.A) and pointwise R - weak commutativity there, without the assumption of completeness of the space and the continuity of the mappings. Our result generalize and extend the results of Pant *et.al* [14] and others.

2. MAIN RESULTS

Now we state our main theorem.

Theorem 2.1: Let f and g be pointwise R-weakly commuting selfmappings of a metric space (X,d) satisfying the property (E.A) and the following conditions hold:

(2.1) $\overline{fX} \subseteq gX$, where $\overline{fX}$ is the closure of the range of f

and

(2.2) $\int_0^{d(fx,fy)} \varphi(t)dt \le k\int_0^{[d(fx,gx)+d(gx,gy)+d(gy,fy)]} \varphi(t)dt$, where $o \le k < 1$, $\varphi \in \Phi$ and satisfy the condition (1.1).

Then f and g have a unique common fixed point.

Proof: Since f and g satisfy the property (E.A), then there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$. Since $t \in \overline{fX}$ and $\overline{fX} \subset gX$ then there exists a $u \in X$ such that $t = gu$. Using (2.2) we get

$$\int_0^{d(fx_n,fu)} \varphi(t)dt \le k\int_0^{[d(fx_n,gx_n)+d(gx_n,gu)+d(gu,fu)]} \varphi(t)dt.$$

By taking the limit, as $n \to \infty$ we have

$$\int_0^{d(fu,gu)} \varphi(t)dt \le k\int_0^{[0+0+d(gu,fu)]} \varphi(t)dt = k\int_0^{d(gu,fu)} \varphi(t)dt.$$

So

$$(1-k)\int_0^{d(fu,gu)} \varphi(t)dt = 0.$$

Since $(1-k) \ne 0$, then using (1.1) we get $fu = gu$. Pointwise $R-$weak commutativity of f and g implies for $R > 0$ that

$$d(fgu, gfu) \le Rd(fu, gu).$$

So

$$\int_0^{d(fgu,gfu)} \varphi(t)dt \le \int_0^{Rd(fu,gu)} \varphi(t)dt = 0.$$

Using (1.1) we have $fgu = gfu$. Also $ffu = fgu = gfu = ggu$. Using (2.2) again we get

$$\int_0^{d(fu,ffu)} \varphi(t)dt \le k\int_0^{[d(fu,gu)+d(gu,gfu)+d(gfu,ffu)]} \varphi(t)dt$$

$$= k\int_0^{d(fu,ffu)} \varphi(t)dt$$

$$(1-k)\int_0^{d(fu,ffu)} \varphi(t)dt = 0.$$

Since $(1-k) \ne 0$, then using (1.1) we get $fu = ffu$. Hence $fu = fgu = gfu$ and fu is a common fixed point of f and g. Now to show the uniqueness of the common fixed point of f and g, let f and g have two distinct common fixed point v and w. Using (2.2) we have

$$\int_0^{d(v,w)} \varphi(t)dt = \int_0^{d(fv,fw)} \varphi(t)dt \le k\int_0^{[d(fv,gv)+d(gv,gw)+d(gw,fw)]} \varphi(t)dt = k\int_0^{d(v,w)} \varphi(t)dt,$$

Since $k < 1$ so we have a contradiction. Thus $v = w$.

Now we give an example to illustrate the above theorem.

Example 2.1: Let $X = [0, 1]$ and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = \begin{cases} 0 & \text{if} \quad x = 0 \\ \dfrac{1}{2n} & \text{if} \quad \dfrac{1}{n+1} < x \le \dfrac{1}{n}, n \in N \end{cases}$$

$$gx = \begin{cases} 0 & \text{if} \quad x = 0 \\ \dfrac{1}{n} & \text{if} \quad \dfrac{1}{n+1} < x \le \dfrac{1}{n}, n \in N \end{cases}.$$

Now to check the property(E.A) we consider the sequence $\{y_n\} = \left\{\frac{1}{n} : n \in N\right\}$, which satisfies

$$\lim_{n\to\infty} d(fy_n, 0) = \lim_{n\to\infty} d(\frac{1}{2n}, 0) = \lim_{n\to\infty} \frac{1}{2n} = 0$$

and $$\lim_{n\to\infty} d(gy_n, 0) = \lim_{n\to\infty} d(\frac{1}{n}, 0) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Hence $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = 0 \in X$ so the pair (f, g) satisfies property (E.A). However one can see that $\lim_{n\to\infty} fgy_n = \lim_{n\to\infty} gfy_n$ so the pair (f, g) is not noncompatible. Also $fX = \{0\} \cup \left\{\frac{1}{2n} : n \in N\right\}$ and $gX = \{0\} \cup \left\{\frac{1}{n} : n \in N\right\}$ and $\overline{fX} = \{0\} \cup \left\{\frac{1}{2n} : n \in N\right\}$. Thus $\overline{fX} \subset gX$.

Define a map $\varphi \in \Phi$ by $\varphi(t) = 3t^2$ for $t > 0$ and $\varphi(0) = 0$, where φ satisfies the condition (1.1). Then for any $\tau > 0$,

$$\int_0^{\tau} \varphi(t)dt = \tau^3.$$

Now we consider the following four cases.

Case-I: If $x = 0$ and $y \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ then

$$\begin{aligned} d(fx, gx) + d(gx, gy) + d(gy, fy) &= 0 + \frac{1}{n} + \left|\frac{1}{n} - \frac{1}{2n}\right| \\ &= \frac{1}{n} + \frac{1}{2n} \\ &= \frac{3}{2n} \end{aligned}$$

$$d(fx, fy) = \frac{1}{2n}.$$

So we have

$$\int_0^{d(fx,fy)} \varphi(t)dt = \left[\frac{1}{2n}\right]^3 = \left[\frac{1}{3}\frac{3}{2n}\right]^3$$
$$= \frac{1}{27}\int_0^{d(fx,gx)+d(gx,gy)+d(gy,fy)} \varphi(t)dt.$$

Case-II: If $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ and $y \in \left(\frac{1}{m+1}, \frac{1}{m}\right]$, $m > n$, then

$$d(fx,gx) + d(gx,gy) + d(gy,fy) = \left|\frac{1}{2n} - \frac{1}{n}\right| + \left|\frac{1}{n} - \frac{1}{m}\right| + \left|\frac{1}{m} - \frac{1}{2m}\right|$$

$$d(fx,fy) = \left|\frac{1}{2n} - \frac{1}{2m}\right|$$
$$= \frac{m-n}{2mn}.$$

So

$$\int_0^{d(fx,fy)} \varphi(t)dt = \left[\frac{m-n}{2mn}\right]^3 = \left[\frac{3m-n}{2mn}.\frac{m-n}{3m-n}\right]^3$$
$$= \left[\frac{m-n}{3m-n}\right]^3 \int_0^{d(fx,gx)+d(gx,gy)+d(gy,fy)} \varphi(t)dt.$$

Case-III: If $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ and $y \in \left(\frac{1}{m+1}, \frac{1}{m}\right]$, $m < n$ then similarly as in Case (ii) we have

$$\int_0^{d(fx,fy)} \varphi(t)dt = \left[\frac{n-m}{2mn}\right]^3 = \left[\frac{3n-m}{2mn}.\frac{n-m}{3n-m}\right]^3$$
$$= \left[\frac{n-m}{3n-m}\right]^3 \int_0^{d(fx,gx)+d(gx,gy)+d(gy,fy)} \varphi(t)dt.$$

Case-IV: If $x, y \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$, then

$$d(fx,gx) + d(gx,gy) + d(gy,fy) = \left|\frac{1}{2n} - \frac{1}{n}\right| + 0 + \left|\frac{1}{n} - \frac{1}{2n}\right|$$
$$= \frac{1}{2n} + \frac{1}{2n}$$
$$= \frac{1}{n}$$

$$d(fx,fy) = 0.$$

So

$$\int_0^{d(fx,fy)} \varphi(t)dt = 0 < \int_0^{d(fx,gx)+d(gx,gy)+d(gy,fy)} \varphi(t)dt\ .$$

Hence in all the above cases there exists a $0 \le k = \max\left\{\frac{1}{27}, \left[\frac{m-n}{3m-n}\right]^3 or \left[\frac{n-m}{3n-m}\right]^3\right\} < 1$ for which the contractive condition (2.2) is satisfied. Thus all the conditions of the above theorem are satisfied and 0 is the unique fixed point of f and g .

In the next theorem we further improve Theorem 1.1 by replacing condition (2.2) with much general inequality and using the strict triangle inequality of integral type.

Theorem 2.2: Let f and g be noncompatible pointwise R-weakly commuing selfmappings of a metric space (X, d) satisfying the property (E.A), (2.1) and the following conditions hold:

(2.3) $$\int_0^{d(fx,fy)} \varphi(t)dt \le a\int_0^{d(fx,gx)} \varphi(t)dt + b\int_0^{d(gx,gy)} \varphi(t)dt + c\int_0^{d(gy,fy)} \varphi(t)dt,$$
$$0 \le a, c < 1, b \ge 0,$$

(2.4) $$\int_0^{d(fx,f^2x)} \varphi(t)dt < \int_0^{d(fx,gx)+d(gx,gfx)+d(gfx,f^2x)} \varphi(t)dt,$$

where $\varphi \in \Phi$ and satisfy the condition (1.1).

Whenever the right hand side is nonzero, then f and g have a common fixed point.

Proof: Since f and g satisfy the property (E.A), then there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$. Since $t \in \overline{fX}$ and $\overline{fX} \subset gX$, then there exists $u \in X$ such that $t = gu$. Using (2.3) we get

$$\int_0^{d(fx_n,fu)} \varphi(t)dt \le a\int_0^{d(fx_n,gx_n)} \varphi(t)dt + b\int_0^{d(gx_n,gu)} \varphi(t)dt + c\int_0^{d(gu,fu)} \varphi(t)dt.$$

By taking the limit, as $n \to \infty$ we have

$$\int_0^{d(gu,fu)} \varphi(t)dt \le 0 + 0 + c\int_0^{d(gu,fu)} \varphi(t)dt = c\int_0^{d(gu,fu)} \varphi(t)dt.$$

So

$$(1-c)\int_0^{d(fu,gu)} \varphi(t)dt = 0.$$

From (1.1) we get $fu = gu$. Pointwise $R-$weak commutativity of f and g implies for $R > 0$, that

$$d(fgu, gfu) \le Rd(fu, gu).$$

So

$$\int_0^{d(fgu,gfu)} \varphi(t)dt \le \int_0^{Rd(fu,gu)} \varphi(t)dt = 0.$$

Thus $fgu = gfu$. Also $ffu = fgu = gfu = ggu$. Using (2.4) again we get

$$\int_0^{d(fu,ffu)} \varphi(t)dt < \int_0^{[d(fu,gu)+d(gu,gfu)+d(gfu,ffu)]} \varphi(t)dt$$
$$= c\int_0^{[0+d(fu,ffu)+0]} \varphi(t)dt$$

$$(1-c)\int_0^{d(fu,ffu)} \varphi(t)dt = 0.$$

Thus from (1.1) we get $fu = ffu$. Hence $fu = fgu = gfu$ and fu is a common fixed point of f and g. This completes the proof.

Remark 2.1: If we take $\varphi(t) = t$ in Theorem 2.1-2.2 we get the results of Pant *et.al* [12], which extend the results of Rhoades [7].

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