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ON QUATERNION-k-NORMAL MATRICES

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ABSTRACT

The concept of quaternion- k-normal (q-k-normal) matrices is introduced. Some basic theorems of q-k-normal and q-k-unitary matrices are discussed.

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Key words*: Quaternion matrix, q-k-normal matrix and q-k-unitary matrix.*

1. INTRODUCTION

Let H be the set of all quaternion numbers. Let $H_{n\times n}$ be the set of all quaternion square matrix over H called by quaternion matrix [6], for a matrix $A\in H_{n\times n}$. $\overline{A}, A^T, A^*, A^{-1}$ and $A^\dagger$ denotes conjugate, transpose, Conjugate transpose, inverse and Moore-Penrose inverse of A respectively. Let k be a fixed product of disjoint transpositions in $S_n=\{1,2,3...n\}$ and let K be the permutation matrix associated with k. The concept of q-k-normal matrices is introduced as generalization of q-k-real and q-k-hermitian and q-k-normal matrices [3, 5]. The q-k-unitary is also discussed in this paper. Clearly K satisfies $K^2=I$, $K=K^T=K^*=KI$.

2. DEFINITIONS AND SOME THEOREMS

Definition 2.1: A matrix $A\in H_{n\times n}$ is said to be quaternion-k-normal denoted by q-k-normal if $AKA^*K=KA^*KA$ where K is a permutation matrix associated with $k(x)$ in S_n.

Example 2.2: A = $\begin{bmatrix} 6+2i & 3 \\ 2 & 6+4i \end{bmatrix}$ is an q-k-normal matrix

Theorem 1: Let $A,B\in H_{n\times n}$. If A and B are q-k-normal with $AKB^*K=KB^*KA$ and $BKA^*K=KA^*KB$ then $A+B$ is q-k-normal.

Proof:

$$\begin{aligned}(A+B)\left[K(A+B)^*K\right] &= (A+B)K\left(A^*+B^*\right)K \\ &= (A+B)\left(KA^*K+KB^*K\right) \\ &= AKA^*K+AKB^*K+BKA^*K+BKB^*K \\ &= \left(KA^*K\right)A+\left(KB^*K\right)A+\left(KA^*K\right)B+\left(KB^*K\right)B\end{aligned}$$

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$$= \left(KA^*K + KB^*K\right)A + \left(KA^*K + KB^*K\right)B$$
$$= \left(KA^*K + KB^*K\right)(A+B)$$
$$= \left[K\left(A+B\right)^* K\right](A+B)$$
$$(A+B)\left[K\left(A+B\right)^* K\right] = \left[K\left(A+B\right)^* K\right](A+B)$$

Therefore $A+B$ is q-*k*-normal.

Theorem 2: If A is q-*k*-normal then A^{-1} is q-*k*-normal.

Proof:

$$A\left(KA^*K\right) = \left(KA^*K\right)A$$
$$\left(A^{-1}\right)K\left(A^{-1}\right)^* K = A^{-1}K\left(A^*\right)^{-1} K$$
$$= A^{-1}K^{-1}\left(A^*\right)^{-1} K^{-1} \left[\because K = K^{-1}\right]$$
$$= \left(AKA^*K\right)^{-1}$$
$$= K^{-1}\left(A^*\right)^{-1} K^{-1}A^{-1}$$
$$= K\left(A^{-1}\right)^* K\left(A^{-1}\right)$$
$$\left(A^{-1}\right)K\left(A^{-1}\right)^* K = K\left(A^{-1}\right)^* K\left(A^{-1}\right)$$

Therefore A^{-1} is q-*k*-normal.

Theorem 3: If A is q-*k*-normal then A^T is q-*k*-normal.

Proof:

$$A\left(KA^*K\right) = \left(KA^*K\right)A$$
$$A^T K\left(A^T\right)^* K = A^T K\left(A^*\right)^T K$$
$$= A^T K^T \left(A^*\right)^T K^T \left[\because K = K^T\right]$$
$$= \left(AKA^*K\right)^T$$
$$= K^T \left(A^*\right)^T K^T \left(A^T\right)$$
$$= K^T \left(A^T\right)^* K^T \left(A^T\right)$$
$$\left(A^T\right)K\left(A^T\right)^* K = K\left(A^T\right)^* K\left(A^T\right)$$

Therefore A^T is q-*k*-normal.

Theorem 4: If A is q-*k*-normal then A^* is q-*k*-normal.

Proof:

$$A\left(KA^*K\right) = \left(KA^*K\right)A$$
$$\left(A^*\right)K\left(A^*\right)^* K = \left(A^*\right)K^*\left(A^*\right)^* K^* \quad \left[\because K = K^*\right]$$
$$= \left(AKA^*K\right)^*$$
$$= K^*\left(A^*\right)^* K^* A^*$$

$$= K\left(A^*\right)^* K\left(A^*\right)$$

$$\left(A^*\right)K\left(A^*\right)^* K = K\left(A^*\right)^* K\left(A^*\right)$$

Therefore A^* is q-*k*-normal.

Theorem 5: If A is q-*k*-normal then A^2 is q-*k*-normal.

Proof:

$$A\left(KA^*K\right) = \left(KA^*K\right)A$$

$$A^2\left[K\left(A^2\right)^* K\right] = A^2K\left(A^*\right)^2 K$$

$$= AA\left(KA^*K\right)\left(KA^*K\right)$$

$$= A\left(KA^*K\right)A\left(KA^*K\right)$$

$$= \left(KA^*K\right)A\left(KA^*K\right)A$$

$$= \left(KA^*K\right)\left(KA^*K\right)AA$$

$$= \left[K\left(A^*\right)^2 K\right]\left(A^2\right)$$

$$= K\left(A^2\right)^* K\left(A^2\right)$$

$$\left(A^2\right)K\left(A^2\right)^* K = K\left(A^2\right)^* K\left(A^2\right)$$

Therefore A^2 is q- *k* -normal.

Theorem 6: If A is q-*k*-normal then A^t is q-*k*-normal.

Proof:

$$AKA^*K = KA^*KA$$

$$A^tK\left(A^t\right)^* K = \left(AAA....A \; t \; times\right)K\left(A^*A^*A^*...A^* \; t \; times\right)K$$

$$= AAA...A\left(KA^*K\right)\left(KA^*K\right)....\left(KA^*K\right)$$

Therefore A^t is q-*k*-normal.

Theorem 7: If A is q-*k*-normal then $A^\dagger$ is q-*k*-normal.

Proof:

$$A\left(KA^*K\right) = \left(KA^*K\right)A$$

$$\left(A^\dagger\right)K\left(A^\dagger\right)^* K = A^\dagger K^\dagger\left(A^*\right)^\dagger K^\dagger \quad \left[\because K = K^\dagger\right]$$

$$= \left(AKA^*K\right)^\dagger$$

$$= K^\dagger\left(A^*\right)^\dagger K^\dagger A^\dagger$$

$$= K\left(A^*\right)^\dagger KA^\dagger$$

$$= K\left(A^\dagger\right)^* K\left(A^\dagger\right)$$

$$\left(A^\dagger\right)K\left(A^\dagger\right)^* K = K\left(A^\dagger\right)^* K\left(A^\dagger\right)$$

Therefore $A^\dagger$ is q-*k*-normal.

Theorem 8: If A is q-*k*-normal then αA is q-*k*-normal.

Proof:

$$A\left(KA^*K\right)=\left(KA^*K\right)A$$

$$\begin{aligned}(\alpha A)K(\alpha A)^*K &= \alpha AK\overline{\alpha}A^*K\\ &= \alpha\overline{\alpha}\left(AKA^*K\right)\\ &= \alpha\overline{\alpha}\left(KA^*KA\right)\\ &= K\alpha\overline{\alpha}A^*KA\\ &= K\left(\overline{\alpha}A^*\right)K(\alpha A)\\ &= K(\alpha A)^*K(\alpha A)\end{aligned}$$

$$(\alpha A)K(\alpha A)^*K = K(\alpha A)^*K(\alpha A)$$

Therefore αA is q-*k*-normal.

Theorem 9: Let A and B are q-*k*-normal in $H_{n\times n}$ then AB is q-*k*-normal if $A\left(KB^*K\right)=\left(KB^*K\right)A$ and $B\left(KA^*K\right)=\left(KA^*K\right)B.$

Proof:

$$A\left(KA^*K\right)=\left(KA^*K\right)A$$

$$\begin{aligned}(AB)\left[K(AB)^*K\right] &= (AB)KB^*A^*K\\ &= (AB)\left(KB^*K\right)\left(KA^*K\right)\\ &= A\left(KB^*KB\right)KA^*K\\ &= \left(KB^*K\right)A\left(KA^*K\right)B\\ &= \left(KB^*K\right)\left(KA^*K\right)AB\\ &= \left(KB^*A^*K\right)(AB)\\ &= \left[K(AB)^*K\right](AB)\end{aligned}$$

$$(AB)\left[K(AB)^*K\right]==\left[K(AB)^*K\right](AB)$$

Therefore AB is q-*k*-normal.

Definition 2.3: A matrix $A\in H_{n\times n}$ is said to be quaternion-*k*-unitary (q-*k*-unitary) if $AKA^*K = KA^*KA = I.$

Example 2.4: A= $\begin{bmatrix}\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{bmatrix}$ is an q-*k*-unitary matrix.

Definition 2.5: Let $A,B\in H_{n\times n}$. The matrix B is said to be quaternion-*k*-unitarily equivalent (q-*k*-unitarily equivalent) to A if there exists an q-*k*-unitary matrix U such that $B = KU^*KAU.$

Example 2.6: Let $A=\begin{bmatrix} 1+i & 2i \\ 3+2i & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 2+2i & 2+3i \\ -2+2i & -3+2i \end{bmatrix}$

Then if we take $U=\begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$ it can be verified that $UKU^*K=KU^*KU=I$ and $B=KU^*KAU$

Hence B is q-k-unitarily equivalent to A.

3. EQUIVALENT CONDITIONS ON Q-$\mathcal{K}$-NORMAL MATRICES

Theorem 3.1: Let $A\in H_{n\times n}$. If A is q-k-unitarily equivalent to a diagonal matrix, then A is q-k- normal.

Proof: Let $A\in H_{n\times n}$. If A is q-k-unitarily equivalent to a diagonal matrix D, then there exists an q-k-unitary matrix P such that $KP^*KAP=D$ Which implies that

$$PKP^*KAPKP^*K=PDKP^*K$$
$$A=PDKP^*K \quad \text{as} \quad PKP^*K=I$$
$$A^*=\left(PDKP^*K\right)^*$$
$$=K^*PK^*D^*P^*=KPKD^*P^*$$

Now
$$\begin{aligned} AKA^*K &=\left(PDKP^*K\right)K\left(KPKD^*P^*\right)K \\ &=PDKP^*K\left(KK\right)PKD^*P^*K \\ &=PDKP^*K\left(K^2\right)PKD^*P^*K \quad \left[\because K^2=I\right] \\ &=PD\left(KP^*KP\right)KD^*P^*K \quad \left[\because KP^*KP=I\right] \\ &=PDIKD^*P^*K \\ &=PDKD^*P^*K \\ &=PD\left(KD^*K\right)\left(KP^*K\right) \end{aligned}$$

$$\begin{aligned} KA^*KA &=K\left(KPKD^*P^*\right)K\left(PDKP^*K\right) \\ &=K^2PKD^*\left(KK\right)P^*KPDKP^*K \\ &=PKD^*K\left(KP^*KP\right)DKP^*K \\ &=P\left(KD^*K\right)DKP^*K \\ &=P(KD^*K)D(KP^*K) \end{aligned}$$

Therefore D and KD^*K are each diagonal $D(KD^*K)=\left(KD^*K\right)D$ and hence $AKA^*K=KA^*KA$ so that A is q-k-normal.

Theorem 3.2: If A is q-k-hermitian then $A^{-1}KA^*K$ is q-k-unitary.

Proof: $A^{-1}\left(KA^*K\right)K\left(A^{-1}KA^*K\right)^*K=A^{-1}AK\left(A^{-1}A\right)^*K$
$$=IK\ I^*K$$
$$=I.$$

Theorem 3.3: If A is q-*k*-normal then $A^{-1}KA^{*}K$ is q-*k*-unitary.

Proof:
$$\begin{aligned}\left(A^{-1}KA^{*}K\right)K\left(A^{-1}KA^{*}K\right)^{*}K &= A^{-1}KA^{*}\left(KK\right)K^{*}\left(A^{*}\right)^{*}K^{*}\left(A^{-1}\right)^{*}K\\ &= A^{-1}\left(KA^{*}KA\right)K^{*}\left(A^{-1}\right)^{*}K\\ &= A^{-1}AKA^{*}KK\left(A^{-1}\right)^{*}K\\ &= KA^{*}\left(A^{-1}\right)^{*}K\\ &= K\left(A^{-1}A\right)^{*}K\\ &= K\left(I^{*}\right)K\\ &= I.\end{aligned}$$

Remark 3.4: From, theorem 3.2 and 3.3 if A is either q-*k*-hermitian or q-*k*-normal then $A^{-1}KA^{*}K$ is q-*k*-unitary.

Theorem 3.5: Let $H, N \in H_{n\times n}$ be invertible. If $B = HNH,$ where H is q-*k*-hermitian and N is q-*k*-normal then $B^{-1}KB^{*}K$ is similar to an q-*k*- unitary matrix.

Proof: Let $H, N \in H_{n\times n}$ then be invertible. If $B = HNH$ then

$$\begin{aligned}B^{-1}KB^{*}K &= \left(HNH\right)^{-1}K\left(HNH\right)^{*}K\\ &= H^{-1}N^{-1}H^{-1}KH^{*}N^{*}H^{*}K\\ &= H^{-1}N^{-1}H^{-1}KH^{*}\left(KK\right)N^{*}H^{*}K\\ &= H^{-1}N^{-1}H^{-1}\left(KH^{*}K\right)KN^{*}H^{*}K \quad [\because \text{H is q-}k\text{-hermitian}]\\ &= H^{-1}N^{-1}\left(H^{-1}H\right)KN^{*}H^{*}K \quad \left[H = KH^{*}K\right]\\ &= H^{-1}N^{-1}KN^{*}H^{*}K\\ &= H^{-1}N^{-1}KN^{*}KKH^{*}K\\ &= H^{-1}N^{-1}\left(KN^{*}K\right)\left(KH^{*}K\right)\end{aligned}$$

Since N is q-*k*- normal from remark 3.4 $N^{-1}KN^{*}K$ is q-*k*- unitary and hence the result follows.

Theorem 3.6: If A is q-*k*-normal and $AB = 0$ then $KA^{*}KB = 0$.

Proof:

$$\begin{aligned}A\left(KA^{*}K\right) &= \left(KA^{*}K\right)A\\ A\left(KA^{*}K\right)B &= \left(KA^{*}K\right)AB\\ &= \left(KA^{*}K\right)0 \quad [\because AB = 0]\\ &= 0\\ A\left(KA^{*}K\right)B &= 0\end{aligned}$$

Therefore $KA^{*}KB = 0 \quad [\because A \neq 0].$

Theorem 3.7: If X is an q-*k*-eigenvector of an q-*k*-normal matrix A corresponding to an q-*k*-eigenvalue λ , then X is also an q-*k*-eigenvector of $KA^{*}K$ corresponding to the q-*k*-eigenvalue $\overline{\lambda}$.

Proof: Let $A \in H_{n\times n}$ be q-k-normal. Since X is an q-k-eigenvector of A corresponding to an q-k-eigenvalue λ, $AX = \lambda X$. Since A is q-k-normal, it can be easily seen that $A-\lambda I$ and $K\left(A-\lambda I\right)^* K$ commute and hence $A-\lambda I$ is q-k-normal. Now

$$AX = \lambda X \Rightarrow \left(A-\lambda I\right)X = 0.$$

Since $A-\lambda I$ is q- k - normal by the above theorem 3.6

$$\left[K\left(A-\lambda I\right)^* K\right]X = 0 \Rightarrow \left[K\left(A^*-\overline{\lambda} I\right)K\right]X = 0$$

$$\left[KA^*K - K\overline{\lambda}IK\right]X = 0$$

$$\left(KA^*K\right)X = K\overline{\lambda}KX$$

$$\left(KA^*K\right)X = \overline{\lambda}KKX$$

$$\left(KA^*K\right)X = \overline{\lambda}X$$

Which leads to the result.

Theorem 3.8: If $A \in H_{n\times n}$ is q-k-unitary and λ is an q-k-eigenvalue of A, then $|\lambda| = 1.$

Proof: Since $A \in H_{n\times n}$ is q-k-unitary, A is q- k - normal. Since λ is an q-k-eigenvalue of A, there exists an q-k-eigenvector $V \neq 0$ such that $AV = \lambda V$ which implies $KA^*KV = \overline{\lambda}V$ as A is q-k-normal.

Now $V = IV = KA^*KAV$ which leads to

$$V - \left[KA^*KA\right]V = 0$$

$$V\left[1 - KA^*KA\right] = 0$$

$$V\left[1 - AKA^*K\right] = 0$$

$$V\left[1 - \lambda\overline{\lambda}\right] = 0$$

Since $V \neq 0$, $1-\lambda\overline{\lambda} = 0$

$$\Rightarrow |\lambda| = 1.$$

Theorem 3.9: Let $A \in H_{n\times n}$. Assume that $A = XP$, where X is q-k-unitary and P is non –singular and q-k-hermition such that if P^2 commutes with X , then P also commutes with X . Then the following conditions are equivalent.

(i) A is q- k-normal
(ii) $XP = PX$
(iii) $AX = VX$
(iv) $AP = PA$

Proof: Let $A = XP$. Since X is q-k-unitary $XKX^*K = KX^*KX = I$ and since P is q-k-hermitian $KP^*K = P$.

$(i) \Leftrightarrow (ii)$ If A is q-k-normal then $AKA^*K = KA^*KA$

Since $A = XP$. $\left(XP\right)K\left(XP\right)^* K = K\left(XP\right)^* K\left(XP\right)$

$$XPKP^*X^*K = KP^*X^*KXP$$

$$XPKP^*\left(KK\right)X^*K = KP^*KKX^*KXP$$

$$XP\left(KP^*K\right)KX^*K = P\left(KX^*KX\right)P$$

$$XPPKX^{*}K = P^{2} \quad \left[\because KX^{*}KX = I\right]$$
$$XP^{2}KX^{*}K = P^{2}$$

Post multiply by X,

We have $XP^{2}\left(KX^{*}KX\right) = P^{2}X$

$$XP^{2} = P^{2}X$$

and hence $XP = PX$ by our assumption.

Conversely, if $XP = PX$ then $\left(KP^{*}K\right)\left(KX^{*}K\right) = \left(KX^{*}K\right)\left(KP^{*}K\right)$

$$\Rightarrow KP^{*}X^{*}K = KX^{*}P^{*}K$$
$$\Rightarrow P^{*}X^{*} = X^{*}P^{*}$$

Now $AKA^{*}K = \left(XP\right)K\left(XP\right)^{*}K$

$$\begin{aligned}
&= XPKP^{*}X^{*}K \\
&= XPKX^{*}P^{*}K \\
&= XP\left(KX^{*}K\right)\left(KP^{*}K\right) \\
&= X\left(KP^{*}K\right)\left(KX^{*}K\right)\left(KP^{*}K\right) \qquad \left[\because P = KP^{*}K\right] \\
&= \left(KP^{*}K\right)\left(KX^{*}K\right)X\left(KP^{*}K\right) \\
&= \left(KP^{*}X^{*}K\right)XP \\
&= K\left(XP\right)^{*}K\left(XP\right)
\end{aligned}$$

$(i) \Leftrightarrow (iv)$: If A is q-*k*-normal, then $AP = \left(XP\right)P$

$$= PXP$$
$$= PA,$$

Conversely, if $AP = PA,$ then $\left(XP\right)P = P\left(XP\right)$ post multiply by P^{-1},

We have $XP = PX$ and so that A is q-*k*-normal.

Theorem 3.10: Let $A \in H_{n\times n}$. Assume that $A = XP$, where X is q-*k*-unitary and A is non singular and q-*k*-hermition such that if P^{2} commutes with X then A also commutes with X. Then the following conditions are equivalent.

(i) A is q-*k*-normal.
(ii) Any q-*k*-eigenvector of X is an q-*k*-eigenvector of A (as long as X has distinct q-*k*-eigenvalues).
(iii) Any q-*k*-eigenvector of A is an q-*k*-eigenvector of X (as long as A has distinct q-*k*-eigenvalues).
(iv) Any q-*k*-eigenvector of X is an q-*k*-eigenvector of A (as long as X has distinct q-*k*-eigenvalues).
(v) Any q-*k*-eigenvector of A is an q- *k*-eigenvector of X (as long as A has distinct q-*k*-eigenvalues).
(vi) Any q- *k*-eigenvector of P is an q-*k*-eigenvector of A (as long as A has distinct q-*k*-eigenvalues).
(vii) Any q- *k*-eigenvector of A is an q-*k*-eigenvector of P (as long as A has distinct q-*k*-eigenvalues)..

Proof:
$(i) \Leftrightarrow (ii)$: Let X have distinct q-*k*-eigenvalues. If we prove $XP = PX \Leftrightarrow$ any q-*k*-eigenvector of X is an q-*k*-eigenvector of $P,$ then $(i) \Leftrightarrow (ii)$ follows by theorem 3.7. Assume that any q-*k*-eigenvector of X is an

q-k-eigenvector of P. If Y is an q-k-eigenvector of X, then X is also an q-k-eigenvector of P. Therefore there exists q-k-eigenvalues λ and μ such that $XY = \lambda Y$ and $PY = \mu Y$. Now $XY = \lambda Y$ implies $PXY = PXY = \lambda\mu Y$. Similarly $PY = \mu Y$ implies $XPY = \lambda\mu Y$. Therefore $PXY = XPY \Rightarrow (PX - XP)Y = 0$ which implies as $PX = XP$ as $Y \neq 0$.

Conversely, assume that $XP = PX$. If Y is an q-k-eigenvector of X, then there exists an q-k-eigenvalue λ such that $XY = \lambda Y$. Let μ be an e q-k-eigenvalue of X such that $XY = \mu Y$. $\therefore \lambda \neq \mu$. Now $XP = PX$ implies $(XP - PX)Y = 0$ which shows that $XPY = \lambda PY$. Similarly $XY = \mu Y$ implies $XPY = \mu PY$.

Therefore $\lambda PY = \mu PY \Rightarrow (\lambda - \mu)PY = 0 \Rightarrow PY = 0$ as $\lambda - \mu \neq 0. \therefore PY = 0Y$ and hence Y is an q-k-eigenvector of P corresponding to the q-k-eigenvalue 0. In general, if μ is any q-k-eigenvalue of X, then we can prove that Y is also an q-k-eigenvector of P. Therefore any q-k-eigenvector of X is also an q-k-eigenvector of P.

Similar proof holds for other equivalent conditions.

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