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INTEGRAL PROPERTIES OF ALEPH FUNCTION WITH TWO GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

The aim of the present paper is to discuss integral properties of Aleph function with two general class of polynomials. During the course of finding, here we establish certain double integral relations pertaining to a product involving two general class of polynomials and the Aleph function. For the sake of illustration, we record here some particular cases of our main results.

***Key words**: Aleph function, general class of polynomials, Hermite polynomials, Laguerre polynomials.*

1. INTRODUCTION

The Aleph function ($\aleph$) introduced by Südlend *et al.* [10]. The notation and complete definition is presented here in the following manner in terms of Mellin-Barnes type integrals.

$$\aleph[z] = \aleph^{M,N}_{y_i',\tau_i;r}[z] = \aleph^{M,N}_{x_i',y_i',\tau_i;r}\left[z \left| \begin{matrix} (a_j,A_j)_{1,n}\,[\tau_i(a_{ji},A_{ji})]_{N+1,x_i',r} \\ (b_j,B_j)_{1,M}\,[\tau_i(b_{ji},B_{ji})]_{M+1,y_i',r} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi\omega}\int_L \Omega^{M,N}_{x_i',y_i',\tau_i;r}(s)\, z^{(s)}\, ds \tag{1.1}$$

For all $z \neq 0$, where $\omega = \sqrt{-1}$ and

$$\Omega^{M,N}_{x_i',y_i',\tau_i;r}(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j + B_j s)\prod_{j=1}^{N}\Gamma(1-a_j - A_j s)}{\sum_{i=1}^{r}\tau_i \prod_{j=N+1}^{x_i'}\Gamma(a_{ji}+A_{ji}s)\prod_{j=N+1}^{y_i'}\Gamma(1-b_{ji}-B_{ji}s)} \tag{1.2}$$

where the integration path $L = L_{i\gamma\omega}$, $\gamma \in R$ extends from $\gamma - i\omega$ to $\gamma + i\omega$, and is such that the poles, assumed to be simple, of $\Gamma(1-a_j - A_j s),\ j = 1,\ldots,n$ do not coincide with the pole of $\Gamma(b_j + B_j s),\ j = 1,\ldots,m$ the parameter x_i', y_i' are non-negative integers satisfying

$$0 \le M \le x_i', 1 < M \le y_i', \tau_i > 0 \text{ for } i = 1,\ldots,r$$

the parameters $A_j, B_j, A_{ij}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C.$ The empty product in (1.2) is interpreted as unity.

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The existence conditions for defining integral (1) are given as

$$\phi_\ell > 0, |\arg(z)| < \frac{\pi}{2}\phi_\ell, \qquad \ell = 1,\ldots,r$$

$$\phi_\ell \geq 0, |\arg(z)| < \frac{\pi}{2}\phi_\ell \text{ and } R\{\xi_\ell\} + 1 < 0$$

where

$$\varphi_\ell = \sum_{j=1}^{N} A_j + \sum_{j=1}^{M} B_j - \tau_\ell \left(\sum_{j=N+1}^{x'_\ell} A_{j\ell} + \sum_{j=M+1}^{y'_\ell} B_{j\ell} \right),$$

and

$$\xi_\ell = \sum_{j=1}^{M} b_j - \sum_{j=1}^{N} a_j + \tau_\ell \left(\sum_{j=M+1}^{y'_\ell} b_{j\ell} - \sum_{j=N+1}^{x'_\ell} a_{j\ell} \right) + \frac{1}{2}(x'_\ell - y'_\ell), \ell = 1,\ldots,r.$$

The general class of polynomial defined by Srivastava [9] are defined as

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k\,!} A'_{n,k}\, x^k, \quad n \geq, 0,1,2,\ldots \tag{1.3}$$

and

$$S_{n'}^{m'}[x^2] = \sum_{k=0}^{[n'/m']} \frac{(-n')_{m'k_1}}{k_1\,!} B'_{n',k_1} [x^2]^{k_1} \tag{1.4}$$

2. THE MAIN INTEGRAL

We shall establish the following integral

A.

$$\int_0^1\int_0^1 \left[\frac{(1-x)}{(1-xy)}vy\right]^{\alpha} \left[\frac{(1-y)}{(1-xy)}\right]^{\beta} \left[\frac{(1-xy)}{(1-x)(1-y)}\right].$$

$$S_n^m\left[\frac{(1-x)}{(1\text{-}xy)}vy\right] S_{n'}^{m'}\left[\frac{(1-x)}{(1-xy)}vy\right]^2$$

$$\aleph_{x'_i,y'_i,\tau_i;r}^{M,N}\left[\frac{(1-y)}{(1-xy)}v\right] dx\,dy$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}\, A'_{nk}}{k\,!} \sum_{k=0}^{[n'/m']} \frac{(-n')_{m'k_1}}{k_1\,!} B'_{n'k_1}\, v^{k+2k_1} \Gamma(k+2k_1+\alpha)$$

$$\aleph_{x'_i+1,y'_{i+1},\tau_i;r}^{M,N+1}\left[\left. \begin{matrix} (1-\beta;1),(a_j,A_j)_{1,N},[\tau_i(a_{ji},A_{ji})]_{N+1,x'_i;r} \\ (b_j,B_j)_{1,M},[\tau_i(b_{ji},B_{ji})]_{M+1,y'_i;r}\ \tau_i(1-k-2k_1-\alpha-\beta)_1 \end{matrix} \right| v \right] \tag{2.1}$$

Provided

$$\text{Re}(\alpha+\beta+b_j/\beta_j] > 0, \; [\arg v] < \frac{T\pi}{2}.$$

m and m' are arbitrary positive integer and coefficients $A'_{nk}(n,k \geq 0)$ and $B'_{n',k_1}(n',k_1 \geq 0)$ are arbitrary constants, real or complex.

Proof: We have the following expression:

$$S_n^m\left[\frac{(1-x)}{(1-xy)}vy\right] S_{n'}^{m'}\left[\frac{(1-x)}{(1\text{-}xy)}vy\right]^2 \aleph_{x'_i,y'_i,\tau_i;r}^{M,N}\left[\frac{(1-y)}{(1-xy)}v\right]$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{m,k}}{k\,!} A'_{n,k} \left[\frac{(1-x)}{(1\text{-}xy)}vy\right]^k \sum_{k=0}^{[n'/m']} \frac{(-n')_{m,k}}{k\,!} B'_{n',k} \left[\frac{(1-x)}{(1\text{-}xy)}vy\right]^{2k_1}$$

$$\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=N+1}^{x_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=N+1}^{y_i^{'}}\Gamma(1-b_{ji}^{'}-B_{ji}^{'}s)}\left[\frac{(1-y)}{(1-xy)}v\right]^s ds \tag{2.2}$$

To prove the given integral, we multiply both sides of (2.2) by

$$\left[\frac{(1-x)}{(1-xy)}y\right]^{\alpha}\left[\frac{(1-y)}{(1\text{-}xy)}\right]^{\beta}\left[\frac{(1-xy)}{(1-x)(1-y)}\right]$$

and on integrating with respect to x and y between 0 and 1 for both the variables, we get the desired result by making use of known result [2,p.145].

B. Here we shall establish the following integral

$$\int_0^{\varpi}\int_0^{\varpi}\varphi(u+v)\,u^{\alpha-1}\,v^{\beta-1}S_n^m[u]\,S_{n'}^{m'}[u^2]\aleph_{y_i^{'},\tau_i;r}^{M,N}[v]\,du\,dv$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}^{'}\sum_{k_1=0}^{[n'/m']}\frac{(-n')_{m'k_1}}{k_1!}B_{n'k_1}^{'}\int_0^{\infty}\phi(\xi)\,\xi^{\alpha+\beta+k+2k_1-1}$$

$$\aleph_{x_i^{'}+1,y_i^{'}+1,\tau_i;r}^{M,N}\left[\begin{matrix}(1-\beta;1),(a_j,A_j)_{1,N},[\tau_i(a_{ji},A_{ji})]_{N+1,x_i^{'};r}\\(b_j,B_j)_{1,M},[\tau_i(b_{ji},B_{ji})]_{M+1,y_i;r[\tau_i(1-k-2k_1-\alpha)}\end{matrix}\middle|\,\xi\right]d\xi \tag{2.3}$$

Provided that $\mathrm{Re}(\alpha+\beta+b_j/\beta_j)>0, m$ and m' are arbitrary positive integer and coefficients $A_{n,k}^{'}$ and $B_{n',k_1}^{'}$ $(n,k>0,\ n'\,k_1\geq 0)$ are arbitrary constants, real or complex.

Proof: We have

$$S_n^m[u]\,S_{n'}^{m'}[u]^2\,\aleph_{x_i^{'},y_i^{'},\tau_i;r}^{M,N}(v)$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}^{'}[u]^k\sum_{k_1=0}^{[n'/m']}\frac{(-n')_{m'k_1}}{k_1!}B_{n,k_1}^{'}[u^2]^{k_1}$$

$$.\qquad \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=N+1}^{x_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=N+1}^{y_i^{'}}\Gamma(1-b_{ji}^{'}-B_{ji}^{'}s)}ds \tag{2.4}$$

[putting value of left hand side of above expression from equations (1.2),(1.3),(1.4)] to prove the integral, we multiply both sides of (2.4) by $\varphi(u+v)\,u^{\alpha-1}.v^{\beta-1}$ and integrating with respect to u and v from both sides between 0 to ∞ for both variables, we will get the required result.

3. PARTICULAR CASES

a) By applying the results obtained (2.1), (2.3) to the case of Hermite polynomial [9] and [12] and by setting

$$S_n^2(x)\rightarrow x^{n/2}\,H_n\left[\frac{1}{2\sqrt{x}}\right] \text{ and}$$

$$S_n^2(x^2)\rightarrow x^n\,H_n\left[\frac{1}{2x}\right]$$

Here we have taken $m=m'=2, A_{n,k}^{'}=(-1)^k, B_{n',k_1}^{'}=(-1)^{k_1}$, we have the following consequences

(1) $$\int_0^1\int_0^1\left[\frac{(1-x)}{(1-xy)}y\right]^2\left[\frac{(1-y)}{(1-xy)}\right]^{\beta}\frac{(1-xy)}{(1-x)(1-y)}\left[\frac{(1-x)}{1-xy}\ vy\right]^{n/2}$$

$$\left[\frac{(1-x)}{(1-xy)}vy\right]^n H_n\left[\frac{1}{2\sqrt{\frac{(1-x)}{(1-xy)}vy}}\right]$$

$$H_n\left[\frac{1}{2\left[\frac{(1-x)}{(1-xy)}vy\right]}\right]\aleph^{M,N}_{x_i',y_i',\tau_i;r}\left[\frac{(1-x)}{(1-xy)}\right]dx\,dy$$

$$=\sum_{k=0}^{[n/2]}\frac{(-n)_{2k}}{k\,!}(-1)^k v^k\sum_{k=0}^{[n'/2]}\frac{(-n')_2}{k_1\,!}(-1)^{k_1}(v^2)^{k_1}\Gamma(k+2k_1+\alpha)$$

$$.\aleph^{M,N}_{x_i'+1,y_i'+1,\tau_i;r}\left[\begin{matrix}(1-\beta;1),(a_j,A_j)_{1,N},[\tau_i(a_{ji},A_{ji})]_{N+1},x_i';r\\(b_j,B_j)_{1,M},[\tau_i(b_{ji},B_{ji})]_{M+1,y_i';r}\,\tau_i(1-k-2k_1-\alpha-\beta)_1\end{matrix}\middle|\,v\right]$$

valid under same conditions as obtained from (2.1).

(2) Now taking

$$\int_0^{\omega}\int_0^{\omega}\varphi(u+v)\,u^{\alpha+n/2+n-1}\,v^{\beta-1}H_n\left[\frac{1}{2\sqrt{u}}\right]H_n\left[\frac{1}{2u}\right]\aleph^{M,N}_{x_i',y_i',\tau_i;r}[v]\,du\,dv$$

$$=\sum_{k=0}^{[n/2]}\frac{(-n)_{2k}}{k\,!}(-1)^k\sum_{k=0}^{[n/2]}\frac{(-n)_{2k_1}}{k_1\,!}(-1)^{k_1}\int_0^{\omega}\varphi(z)\,z^{\alpha+\beta+k-2k_1-1}\Gamma(k+2k_1+\alpha)$$

$$.\aleph^{M,N+1}_{x_i'+1,y_i'+1,\tau_i;r}\left[\begin{matrix}(1-\beta;1),(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{N+1},x_i';r\\(b_j,B_j)_{1,M},[\tau_i(b_{ji},B_{ji})]_{M+1,y_i';r}\,\tau_i(1\text{-}k\text{-}2k_1-\alpha-\beta)_1\end{matrix}\middle|\,v\right]$$

valid under the same conditions as required for (2.3).

b) For Laguerre polynomial [9] and [12], setting

$$S_n'(x)\to L_n^{(\alpha')}(x)$$

and

$$S_{n'}^1(x)^2\to L_n^{(\alpha')}(x^2),$$

Here $m=m'=1, A'_{n,k}=\binom{n+\alpha'}{n}\frac{1}{(\alpha'+1)_{k'}}, B'_{n,k}=\binom{n'+\alpha'}{n'}\frac{1}{(\alpha'+1)_{k'}}$

The results obtained (2.1), (2.3) reduces to following expression

(1) $$\int_0^1\int_0^1\left[\frac{(1-x)}{(1-xy)}y\right]^{\alpha}\left[\frac{(1-y)}{(1-xy)}\right]^{\beta}\left[\frac{(1-xy)}{(1-x)(1-y)}\right]L_n^{\alpha'}\left[\frac{(1-x)}{(1\text{-}xy)}vy\right]L_{n'}^{\alpha'}\left[\frac{(1-x)}{1-xy}vy\right]^2\aleph^{M,N}_{x_i',y_i',\tau_i;r}\left[\frac{(1-x)}{(1-xy)}v\right]dx\,dy\,.$$

$$=\sum_{k=0}^{n}\frac{(-n)_k}{k\,!}\binom{n+\alpha'}{n}\frac{1}{(\alpha'+1)_k}v^k\sum_{k_1=0}^{n'}\frac{(-n')_{k_1}}{k_1\,!}\binom{n'+\alpha'}{n'}\frac{1}{(\alpha'+1)_{k'}}(v^2)^{k_1}\Gamma k+2k_1+\alpha$$

$$.\aleph^{M,N+1}_{x_i'+1,y_{i+1}',\tau_i;r}\left[\begin{matrix}(1-\beta;1),(a_j,A_j)_{1,N},[\tau_i(a_{ji},A_{ji})]_{N+1},x_i';r\\(b_j,B_j)_{1,M},[\tau_i(b_{ji},B_{ji})]_{M+1,y_i';r}\,\tau_i(1-2k-\alpha-\beta,1)\end{matrix}\middle|\,v\right]$$

valid under same conditions as required for (2.1).

(2) We have

$$\int_0^{\omega}\int_0^{\omega} \varphi(u+v)\, u^{\alpha-1}\, v^{\beta-1}\, L_n^{(\alpha)}(u)\, L_{n'}^{(\alpha')}(v^2)\, \aleph^{M,N}_{x_i',y_i',\tau_i;r}(v)$$

$$=\sum_{k=0}^{n}\frac{(-n)_k}{k\,!}\binom{n+\alpha'}{n}\frac{1}{(\alpha'+1)_{k'}}\sum_{k_1=0}^{n'}\frac{(-n')_{k_1}}{k_1\,!}\binom{n'+\alpha'}{n'}\int_0^{\omega}\varphi(z)\, z^{\alpha+\beta+3k-1}\Gamma k+2k_1+\alpha$$

$$.\aleph^{M,N+1}_{x_i'+1,y_{i+1}',\tau_i;r}\left[\begin{matrix}(1-\beta;1),(a_j,A_j)_{1,N},[\tau_i(a_{ji},A_{ji})]_{N+1,x_i';r}\\ (b_j,B_j)_{1,M},[\tau_i(b_{ji},B_{ji})]_{M+1,y_i';r}\ \tau_i(1-2k-\alpha-\beta,1)\end{matrix}\right]$$

valid under same conditions as required for (2.3).

CONCLUSION

The results here are basic in nature and are likely to find useful applications in several fields notably electrical networks, statistical mechanics and probability theory.

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