International Journal of Mathematical Archive-7(7), 2016, 53-58
IJMA Available online through www.ijma.info ISSN 2229 – 5046

HERONIAN MEAN DERIVATIVE - BASED CLOSED NEWTON COTES QUADRATURE

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(Received On: 29-06-16; Revised & Accepted On: 19-07-16)

ABSTRACT

A new scheme of numerical integration formula is presented, which uses the Heronian mean at the evaluation of function derivative. These Heronian mean derivative - based closed Newton cotes quadrature rules (HRDCNC) obtain an increase of single order of precision than the existing closed Newton cotes quadrature rule (CNC). This set of rules are derived by using the concept of precision, along with the error terms. Finally, the effectiveness of the proposed rule is verified with a numerical Examples.

Keyword: *Closed Newton-Cotes formula, Error terms, Heronian mean derivative, Numerical examples, Numerical integration.*

AMS Mathematics Subject Classification (2010): *65D30, 65D32.*

1. INTRODUCTION

Numerical integration is an important subfield of numerical analysis and a wide variety of elaborate schemes have been developed for treating many different types of integrals. It has several applications in the field of physics and Engineering. The problem of calculating an integral numerically occurs very often in physics. First, subdividing the integrand over an equidistant mesh and applies a simple rule of integration. Finally, the required accuracy is achieved [7].General quadrature rule for the evaluation of numerical integration is given by

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \tag{1}$$

subdividing the finite interval $a \leq x \leq b$ into a large number of subintervals, by defining (n+1) intermediate points such that $a = x_0 < x_1 < \ldots < x_n = b$, $x_i=x_0+ih$, i=0,1,2,..n, $h = \frac{b-a}{n}$ and (n+1) weights $w_0, w_1, \ldots, w_n$. Select the values for w_i, i=0, 1,...n. so that the error of approximation for the method based on the precision of a quadrature formula is zero, that is

$$E_n[f] = \int_a^b f(x)dx - \sum_{i=0}^{n} w_i f(x_i) = 0, \quad \text{for } f(x) = x^j \quad j = 0,1,\ldots n \tag{2}$$

Definition 1: An integration method of the form (1) is said to be of order P, if it produces exact results ($E_n[f] = 0$) for all polynomials of degree less than or equal to P [10].

The well-known method for the evaluation of numerical integration is closed Newton cotes quadrature formula. we list some of the formula that depend on the integer value of n, are given below

When n=1: Trapezoidal rule

$$\int_a^b f(x)dx = \frac{b-a}{2}(f(a) + f(b)) - \frac{(b-a)^3}{12} f''(\xi), \quad where\ \xi \in (a,b) \tag{3}$$

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When n =2: Simpson's 1/3rd rule

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi),\ \ where\ \xi\ \in (a,b) \tag{4}$$

When n =3: Simpson's 3/8th rule

$$\int_a^b f(x)dx = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480}\ f^{(4)}(\xi),\ where\ \xi \in (a,b) \tag{5}$$

When n=4: Boole's rule

$$\int_a^b f(x)dx = \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right)\ + 7f(b)\right]\ - \frac{(b-a)^7}{1935360} f^{(6)}(\xi),\ where\ \xi \in (a,b) \tag{6}$$

It is known that the degree of precision is n+1 for even value of n and n for odd value of n.

There are so many methods are focused on increasing the order of accuracy of standard numerical integration formula. Dehghan *et al*. improved the closed Newton cotes formula [4] by including the location of boundaries of the interval as two additional parameter, and rescaling the original integral to fit the optimal boundary locations. They have applied this different approach to open, semi-open, Gauss Legendre and Gauss Chebyshev [5, 6, 1, 8] Newton- cotes quadrature rules. Burg *et al.*, and related publications have proposed derivative based closed, open and Midpoint quadrature rules [2, 9, 3]. In their following work, Weijing Zhao and Hongxing Li [14] introduced a Midpoint derivative - based closed Newton-Cotes quadrature rules. Recently, we proposed Midpoint derivative - based open Newton cotes quadrature rule [11], Geometric mean[12] and Harmonic mean[13] derivative based closed Newton cotes quadrature rule.

In this paper, Heronian mean derivative-based closed Newton cotes quadrature formulas are proposed by using the heronian mean value at the computation of derivative. The proposed formula increase the order of accuracy of the existing numerical integration by a single order. The error terms are derived by using the concept of precision and compared with the existing method. Numerical examples are presented, that demonstrate that the order of accuracy of the proposed scheme is higher than the existing method.

2. Heronian mean derivative -based closed Newton cotes quadrature rule

A new set of Heronian mean derivative - based closed Newton cotes formula can be obtained for the evaluation of definite integral within the interval [a, b].

Theorem 2.1: Closed Trapezoidal rule (n=1) using heronian mean derivative is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(a) + f(b)] - \frac{(b-a)^3}{12} f''\left(\frac{a+\sqrt{ab}+b}{3}\right), \tag{7}$$

The precision of this method is 2.

Proof: Since the rule (3) has the degree of precision 1. Now use the rule (7) for $f(x) = x^2$.
When $f(x) = x^2$, $\int_a^b x^2\,dx = \frac{1}{3}(b^3 - a^3)$; $[n = 1] \Longrightarrow \frac{b-a}{2}(a^2 + b^2) - \frac{2(b-a)^3}{12} = \frac{1}{3}(b^3 - a^3)$.

It shows that the solution is exact. Therefore, the precision of closed Trapezoidal rule with heronian mean derivative is 2.

Theorem 2.2: Closed Simpson's 1/3rd rule with heronian Mean derivative (n=2) is

$$\int_a^b f(x)dx\ \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right), \tag{8}$$

The precision of this method is 4.

Proof: Since the rule (4) has the degree of precision 3.Now use the rule (8) for $f(x) = x^4$.
When $f(x) = x^4$, $\int_a^b x^4\,dx = \frac{1}{5}(b^5 - a^5)$; $[n = 2] \Longrightarrow \left(\frac{b-a}{6}\right)\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right] - \frac{24(b-a)^5}{2880} = \frac{1}{5}(b^5 - a^5)$.

It shows that the solution is exact. Therefore, the precision of closed Simpson's1/3rd rule with heronian mean derivative is 4.

Theorem 2.3: Closed Simpson's 3/8rd rule with heronian mean derivative (n=3) is

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right), \tag{9}$$

The precision of this method is 4.

Proof: Since the rule (5) has the degree of precision 3.Now use the rule (9) for $f(x) = x^4$. When $f(x) = x^4$, $\int_a^b x^4\,dx = \frac{1}{5}(b^5 - a^5)$; $[n = 3] \Rightarrow \left(\frac{b-a}{8}\right)\left[a^4 + 3\left(\frac{2a+b}{3}\right)^4 + 3\left(\frac{a+2b}{3}\right)^4 + b^4\right] - \frac{24(b-a)^5}{6480} = \frac{1}{5}(b^5 - a^5)$.

It shows that the solution is exact. Therefore, the precision of closed Simpson's3/8rd rule with heronian mean derivative is 4.

Theorem 2.4: Closed Boole's rule with heronian mean derivative (n=4) is

$$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}\left(\frac{a+\sqrt{ab}+b}{3}\right) \tag{10}$$

The precision of this method is 6.

Proof: Since the rule (6) has the degree of precision 5. Now use the rule (10) for $f(x) = x^6$.
When $f(x) = x^6$, $\int_a^b x^6\,dx = \frac{1}{7}(b^7 - a^7)$;

$$[n = 4] \Rightarrow \left(\frac{b-a}{90}\right)\left[7a^6 + 32\left(\frac{3a+b}{4}\right)^6 + 12\left(\frac{a+b}{2}\right)^6 + 32\left(\frac{a+3b}{4}\right)^6 + 7b^6\right] + \frac{720(b-a)^7}{1935360} = \frac{1}{7}(b^7 - a^7).$$

It shows that the solution is exact. Therefore, the precision of closed Boole's rule with heronian mean derivative is 6.

3. The error terms of Heronian mean derivative –based closed Newton cotes quadrature rule

The error terms are obtained by using the method based on the precision of a quadrature formula. That is, the calculation of the error terms are obtained by using the difference between the quadrature formula for the monomial $\frac{x^{p+1}}{(p+1)!}$ and the exact result $\frac{1}{(p+1)!}\int_a^b x^{p+1}\,dx$ where p is the precision of the quadrature formula.

Theorem 3.1: Heronian mean derivative-based closed Trapezoidal rule (n=1) with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\big(f(a) + f(b)\big) - \frac{(b-a)^3}{12} f''\left(\frac{a+\sqrt{ab}+b}{3}\right) - \frac{(b-a)^3}{72}(\sqrt{b}-\sqrt{a})^2 f^{(3)}(\xi), \tag{11}$$

where $\xi \in (a, b)$.This is fourth order accurate with the error term

$$E_1[f] = -\frac{(b-a)^3}{72}(\sqrt{b}-\sqrt{a})^2 f^{(3)}(\xi).$$

Proof:
Let $f(x) = \frac{x^3}{3!}$, $\frac{1}{3!}\int_a^b x^3\,dx = \frac{1}{24}(b^4 - a^4)$;

$$\frac{b-a}{2}\big(f(a) + f(b)\big) - \frac{(b-a)^3}{12} f''\left(\frac{a+\sqrt{ab}+b}{3}\right) = \frac{b-a}{3!.2}\left(b^3 + a^3 - (b-a)^2\left(\frac{a+\sqrt{ab}+b}{3}\right)\right),$$

Therefore,

$$\frac{1}{24}(b^4 - a^4) - \frac{b-a}{3!.2}\left(b^3 + a^3 - (b-a)^2\left(\frac{a+\sqrt{ab}+b}{3}\right)\right) = -\frac{(b-a)^3}{72}(\sqrt{b} - \sqrt{a})^2.$$

Therefore the error term is,

$$E_1[f] = -\frac{(b-a)^3}{72}(\sqrt{b}-\sqrt{a})^2 f^{(3)}(\xi).$$

Theorem 3.2: Heronian mean derivative-based closed Simpson's1/3rd rule (n=2) with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right) - \frac{(b-a)^5}{17280}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi), \tag{12}$$

where $\xi \in (a, b)$.This is sixth order accurate with the error term

$$E_2[f] = -\frac{(b-a)^5}{17280}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi).$$

Proof:
Let $f(x) = \frac{x^5}{5!}$, $\frac{1}{5!}\int_a^b x^5\,dx = \frac{1}{720}(b^6 - a^6)$;

$$\frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right) = \frac{b-a}{5!.48}\left(8a^5 + (a+b)^5 + 8b^5 - 2(b-a)^4\left(\frac{a+\sqrt{ab}+b}{3}\right)\right),$$

Therefore,

$$\frac{1}{720}(b^6 - a^6) - \frac{b-a}{5!.48}\left(8a^5 + (a+b)^5 + 8b^5 - 2(b-a)^4\left(\frac{a+\sqrt{ab}+b}{3}\right)\right) = -\frac{(b-a)^5}{17280}(\sqrt{b} - \sqrt{a})^2.$$

Therefore the error term is,

$$E_2[f] = -\frac{(b-a)^5}{17280}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi).$$

Theorem 3.3: Heronian mean derivative-based closed Simpson's 3/8th rule (n=3) with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right) - \frac{(b-a)^5}{38880}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi), \quad (13)$$

Where $\xi \in (a, b)$.This is sixth order accurate with the error term

$$E_3[f] = -\frac{(b-a)^5}{38880}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi).$$

Proof:

Let $f(x) = \frac{x^5}{5!}$, $\frac{1}{5!}\int_a^b x^5\,dx = \frac{1}{720}(b^6 - a^6)$;

$$\frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right) = \frac{b-a}{5!.648}\begin{pmatrix} 81a^5 + (2a+b)^5 + (a+2b)^5 + 81b^5 \\ -12(b-a)^4\left(\frac{a+\sqrt{ab}+b}{3}\right)\end{pmatrix},$$

Therefore,

$$\frac{1}{720}(b^6 - a^6) - \frac{b-a}{5!.648}\begin{pmatrix} 81a^5 + (2a+b)^5 + (a+2b)^5 + 81b^5 \\ -12(b-a)^4\left(\frac{a+\sqrt{ab}+b}{3}\right)\end{pmatrix} = -\frac{(b-a)^5}{38880}(\sqrt{b} - \sqrt{a})^2.$$

Therefore the error term is,

$$E_3[f] = -\frac{(b-a)^5}{38880}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi).$$

Theorem 3.4: Heronian mean derivative-based closed Boole's rule (n=4) with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}\left(\frac{a+\sqrt{ab}+b}{3}\right) - \frac{(b-a)^7}{11612160}(\sqrt{b}-\sqrt{a})^2 f^{(7)}(\xi),$$

where $\xi \in (a, b)$.This is eighth order accurate with the error term

$$E_4[f] = -\frac{(b-a)^7}{11612160}(\sqrt{b}-\sqrt{a})^2 f^{(7)}(\xi).$$

Proof: Let $f(x) = \frac{x^7}{7!}$, $\frac{1}{7!}\int_a^b x^7\,dx = \frac{1}{40320}(b^6 - a^6)$; $\frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right]$

$$- \frac{(b-a)^7}{11612160} f^{(6)}\left(\frac{a+\sqrt{ab}+b}{3}\right)$$

$$= \frac{b-a}{7!.768}(97a^7 + 91a^6b + 105a^5b^2 + 91a^4b^3 + 91a^3b^4 + 105a^2b^5 + 91ab^6 + 97b^7 - 2(b-a)^4\left(\frac{a+\sqrt{ab}+b}{3}\right)),$$

Therefore

$$\frac{1}{720}(b^6 - a^6) - \frac{b-a}{7!.768}(97a^7 + 91a^6b + 105a^5b^2 + 91a^4b^3 + 91a^3b^4 + 105a^2b^5 + 91ab^6 + 97b^7 - 2(b-a)^4\left(\frac{a+\sqrt{ab}+b}{3}\right)),$$

$$= -\frac{(b-a)^7}{11612160}(\sqrt{b} - \sqrt{a})^2.$$

Therefore the error term is,

$$E_4[f] = -\frac{(b-a)^7}{11612160}(\sqrt{b}-\sqrt{a})^2 f^{(7)}(\xi).$$

The summary of precision, the orders and the error terms for Heronian mean derivative based closed Newton- Cotes quadrature are shown in Table 1.

Table-1: Comparison of error terms

Rules	Precision	Order	Error terms
Trapezoidal rule (n=1)	2	4	$-\frac{(b-a)^3}{72}(\sqrt{b}-\sqrt{a})^2 f^{(3)}(\xi)$
Simpson's 1/3rd rule (n=2)	4	6	$-\frac{(b-a)^5}{17280}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi)$
Simpson's 3/8th rule (n=3)	4	6	$-\frac{(b-a)^5}{38880}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi).$
Boole's rule (n=4)	6	8	$-\frac{(b-a)^7}{11612160}(\sqrt{b}-\sqrt{a})^2 f^{(7)}(\xi)$

4. NUMERICAL EXAMPLES

In this section, in order to compare the effectiveness of the closed Newton cotes formula and the Heronian mean derivative - based closed Newton cotes formula, the integrals: $\int_1^2 xe^x \, dx$ and $\int_1^2 \sqrt{x} dx$ are evaluated and the results are compared and are shown in Table 2 and 3.

We know that

Error = |Exact value - Approximate value|

Example 1: solve $\int_1^2 xe^x dx$ and compare the solutions with the CNC and HRDCNC rules.

Solution: Exact value of $\int_1^2 xe^x dx$= 7.389056099

Table-2: Comparison of CNC and HRDCNC rules

value of n	CNC		HRDCNC	
	Approximate value	Error	Approximate value	Error
n=1	8.748197013	1.359140914	7.488265778	0.099209679
n=2	7.397754741	0.008698642	7.389480480	0.000424381
n=3	7.392939482	0.003883383	7.389262033	0.000205934
n=4	7.389073662	0.000017563	7.389056840	0.000000741

Example 2: solve $\int_1^2 \sqrt{x} dx$ and compare the solutions with the CNC and HRDCNC rules.

Solution: Exact value of $\int_1^2 \sqrt{x} dx$=1.218951416

Table-3: Comparison of CNC and HRDCNC rules

Value of n	CNC		HRDCNC	
	Approximate value	Error	Approximate value	Error
n=1	1.207106781	0.011844635	1.218779194	0.000172222
n=2	1.218865508	0.000085908	1.218949748	0.000001668
n=3	1.218912315	0.000039101	1.218949755	0.000001661
n=4	1.218950467	0.000000949	1.218951379	0.000000037

5. CONCLUSION

In this paper, a new method of Heronian mean derivative - based closed Newton - Cotes quadrature formulas were presented for the evaluation of definite integral. This proposed formula increase the order of accuracy than the existing formula. The error of approximation for the proposed rule is obtained by using the concept of precision. Finally, numerical examples are solved to show the accuracy of the proposed formula.

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Source of support: Nil, Conflict of interest: None Declared