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SYMMETRIC BI- f -DERIVATIONS IN ALMOST DISTRIBUTIVE LATTICES

G. C. RAO*[1], K. RAVI BABU [2]

[1] Department of Mathematics, Andhra University, India.

[2] Department of Mathematics, Govt. Degree College, Sabbavaram, Visakhapatnam, Andhra Pradesh, India - 530003.

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ABSTRACT

In this paper, we introduce the concept of symmetric bi- f -derivation in an Almost Distributive Lattice (ADL) and derive some important properties of symmetric bi- f -derivations in ADLs.

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1. INTRODUCTION

The concept of derivation in an ADL was introduced in our earlier paper [11]. The notion of derivation in Lattices was first given in G.Szasz [15] in 1974. Earlier Posner[9] introduced derivations in ring theory and later several authors worked on it ([2], [5]). Several authors worked on derivations in Lattices ([1], [3], [4], [6], [7], [8], [16], [17] and [18]). We have introduced the concept of f -derivations in an ADL in our paper [12] and the concept of symmetric bi-derivations in an ADL in our paper [13]. The concept of symmetric bi- f -derivations in lattices was introduced by Kyung Ho Kim [6] in 2012.

In 1980, the concept of an Almost Distributive Lattice(ADL) was introduced by U.M.Swamy and G.C Rao[14]. In this paper, we introduce the concept of symmetric bi- f -derivation in an ADL and derive some important properties. We introduce the concept of an isotone symmetric bi- f -derivation in an ADL and establish a set of conditions which are sufficient for the trace of a symmetric bi- f -derivation on an ADL with a maximal element to become an isotone . Also, we prove $D(x, y) = fx \wedge D(x \vee z, y)$ if D is isotone and $D(x, y) = [fx \wedge D(x \vee z, y)] \vee D(x, y)$ if f is a join homomorphism or an increasing function on L. We define a set $F_a(L)$ for each $a \in L$ and prove that it is a weak ideal if D is a join preserving symmetric bi- f -derivation on an ADL L with 0 where f is a join-homomorphism.

2. PRELIMINARIES

In this section , we recollect certain basic concepts and important results on Almost Distributive Lattices.

Corresponding Author: G. C. Rao[1]*
[1] *Department of Mathematics, Andhra University, India.*

Definition 2.1: [10] An algebra $(L,\vee,\wedge)$ of type $(2,2)$ is called an Almost Distributive Lattice if it satisfies the following axioms:

$L_1:\ (a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c)\ \ (RD\wedge)$

$L_2:\ a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)\ \ (LD\wedge)$

$L_3:\ (a\vee b)\wedge b=b$

$L_4:\ (a\vee b)\wedge a=a$

$L_5:\ a\vee(a\wedge b)=a$

Definition 2.2: [10] Let X be any non-empty set. Define, forany $x,y\in L,\ x\vee y=x$ and $x\wedge y=y.$ Then $(X,\vee,\wedge)$ is an ADL and such an ADL, we call discrete ADL.

Through out this paper L stands for an ADL $(L,\vee,\wedge)$ unless otherwise specified.

Lemma 2.3: [10] For any $a,b\in L,$ we have

$(i)\ \ a\wedge a=a$

$(ii)\ \ a\vee a=a.$

$(iii)\ (a\wedge b)\vee b=b$

$(iv)\ \ a\wedge(a\vee b)=a$

$(v)\ \ a\vee(b\wedge a)=a.$

$(vi)\ \ a\vee b=a$ if and only if $a\wedge b=b$

$(vii)\ a\vee b=b$ if and only if $a\wedge b=a.$

Definition 2.4: [10] For any $a,b\in L,$ we say that a is less than or equal to b and write $a\le b,$ if $a\wedge b=a$ or , equivalently, $a\vee b=b.$

Definition 2.5: [10] For any $a,b,c\in L$, we have the following

(i) The relation $\le$ is a partial ordering on $L.$

(ii) $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c).\ \ (LD\vee)$

(iii) $(a\vee b)\vee a=a\vee b=a\vee(b\vee a).$

(iv) $(a\vee b)\wedge c=(b\vee a)\wedge c.$

(v) The operation $\wedge$ is associative in $L.$

(vi) $a\wedge b\wedge c=b\wedge a\wedge c.$

Theorem 2.6: [10] For any $a,b\in L,$ the following are equivalent.

$(i)\ \ (a\wedge b)\vee a=a$

$(ii)\ \ a\wedge(b\vee a)=a$

$(iii)\ \ (b\wedge a)\vee b=b$

$(iv)\ \ b\wedge(a\vee b)=b$

$(v)\ \ a\wedge b=b\wedge a$

$(vi)\ \ a\vee b=b\vee a$

(vii) the supremum of a and b exists in L and equals to $a\vee b$

$(viii)$ there exists $x\in L$ such that $a\le x$ and $b\le x$

(ix) the infimum of a and b exists in L and equals to $a\wedge b.$

Definition 2.7: [10] L is said to be associative, if the operation $\vee$ in L is associative.

Theorem 2.8: [10] The following are equivalent.

(i) L is a distributive lattice.

(ii) the poset $(L,\leq)$ is directed above.

(iii) $a\wedge(b\vee a)=a,$ for all $a,b\in L.$

(iv) the operation $\vee$ is commutative in $L.$

(v) the operation $\wedge$ is commutative in $L.$

(vi) the relation $\theta:=\{(a,b)\in L\times L\,|\,a\wedge b=b\}$ is anti-symmetric.

(vii) the relation θ defined in (vi) is a partial order on $L.$

Lemma 2.9: [10] For any $a,b,c,d\in L$,we have the following

(i) $a\wedge b\leq b$ and $a\leq a\vee b$

(ii) $a\wedge b=b\wedge a$ whenever $a\leq b$.

(iii) $[a\vee(b\vee c)]\wedge d=[(a\vee b)\vee c]\wedge d.$

(iv) $a\leq b$ implies $a\wedge c\leq b\wedge c,$ $c\wedge a\leq c\wedge b$ and $c\vee a\leq c\vee b.$

Definition 2.10: [10] An element $0\in L$ is called zero element of $L,$ if $0\wedge a=0$ for all $a\in L.$

Lemma 2.11: [10] If L has 0 , then for any $a,b\in L$, we have the following

(i) $a\vee 0=a,$ (ii) $0\vee a=a$ and (iii) $a\wedge 0=0.$

(iv) $a\wedge b=0$ if and only if $b\wedge a=0.$

Definition 2.12: [14] Let L be a non-empty set and $x_0\in L$. Define, for $x,y\in L$,

$$x\wedge y = y \;\; if \;\; x\neq x_0$$
$$= x \;\; if \;\; x=x_0 \quad \text{and}$$
$$x\vee y = x \;\; if \;\; x\neq x_0$$
$$= y \;\; if \;\; x=x_0$$

, then $(L,\vee,\wedge,x_0)$ is an ADL with x_0 as zero element. This is called discrete ADL with zero.

An element $x\in L$ is called maximal if, for any $y\in L$, $x\leq y$ implies $x=y$.

We immediately have the following.

Lemma 2.13: [10] For any $m\in L$, the following are equivalent:

(1) m is maximal

(2) $m\vee x=m$ for all $x\in L$

(3) $m\wedge x=x$ for all $x\in L$.

Definition 2.14: [10] A nonempty subset I of L is said to be an ideal if and only if it satisfies the following:

(1) $a,b\in I\Rightarrow a\vee b\in I$

(2) $a\in I, x\in L\Rightarrow a\wedge x\in I$.

Definition 2.15: [10] A nonempty subset I of L is said to be an initial segment of L if, $a\in L$ and $x\in L$ such that $x\leq a$ imply that $x\in L$.

Definition 2.16: [13] A nonempty subset I of L is said to be a weak ideal if and only if it satisfies the following:

(1) $a,b\in I\Rightarrow a\vee b\in I$

(2) I is an initial segment of L .

Observe that every ideal of L ia weak ideal, but not converse.

Definition 2.17: [10] A function $f : L \to L$ is said to be an ADL homomorphism if it satisfies the following:

(1) $f(x \wedge y) = fx \wedge fy$,

(2) $f(x \vee y) = fx \vee fy$ for all $x, y \in L$.

Definition 2.18: A function $d : L \to L$ is called an isotone, if $dx \le dy$ for any $x, y \in L$ with $x \le y$.

3. SYMMETRIC bi- f -Derivations IN ADLs

We begin this section with the following definition of a symmetric map and a symmetric bi-derivation in an ADL.

Definition 3.1: [13] A mapping $D : L \times L \to L$ is called symmetric if $D(x, y) = D(y, x)$ for all $x, y \in L$.

If $D(x, z) \le D(y, z)$ for any $x, y \in L$ with $x \le y$, then we call D as an isotone map on L.

Definition 3.2: [13] A symmetric function $D : L \times L \to L$ is called a symmetric bi-derivation on L, if $D(x \wedge y, z) = [y \wedge D(x, z)] \vee [x \wedge D(y, z)]$ for all $x, y, z \in L$.

Observe that a symmetric bi-derivation D on L also satsfies

$$D(x, y \wedge z) = [z \wedge D(x, y)] \vee [y \wedge D(x, z)] \text{ for all } x, y, z \in L.$$

The following definition introduces the notion of an symmetric bi- f -derivation on ADLs.

Definition 3.3: A symmetric function $D : L \times L \to L$ is called a symmetric bi- f -derivation on, if there exists afunction $f : L \to L$ such that

$$D(x \wedge y, z) = [fy \wedge D(x, z)] \vee [fx \wedge D(y, z)] \text{ for all } x, y, z \in L.$$

Obviously, a symmetric bi- f -derivation D on L satisfies the relation

$$D(x, y \wedge z) = [fz \wedge D(x, y)] \vee [fy \wedge D(x, z)] \text{ for all } x, y, z \in L.$$

Example 3.4: Let $f : L \to L$ be a function such that $f(x \wedge y) = fx \wedge fy$ for all $x, y \in L$. Let $a \in L$ and define a function $D : L \times L \to L$ by $D(x, y) = fx \wedge fy \wedge a$ for all $x, y \in L$. Then D is a symmetric bi- f -derivation on L.

Example 3.5: Every symmetric bi-derivation on L is a symmetric bi- f -derivation, where $f : L \to L$ is the identity map.

But, a symmetric bi- f -derivation need not be a symmetric bi-derivation. For, consider the following example.

Example 3.6:. Let L be discrete ADL with 0 and $0 \ne a \in L$. Define a function $f : L \to L$ by $fx = a$ for all $x \in L$ and $D : L \times L \to L$ by $D(x, y) = a$ for all $x, y \in L$, then D is a symmetric bi- f -derivation on L but not a symmetric bi-derivation.

Example 3.7: Let L be a discrete ADL with at least two elements. Define a function $D : L \times L \to L$ by $D(x, y) = x \wedge y$ for all $x, y \in L$, then D is not a symmetric bi- f -derivation on L. Since, it is not a symmetric map.

Lemma 3.8: Let D be a symmetric bi- f -derivation on L. Then the following hold:

1. $D(x, y) = fx \wedge D(x, y)$ for all $x, y \in L$
2. $D(x \wedge z, y) = [fx \vee fy] \wedge D(x \wedge z, y)$ for all $x, y, z \in L$
3. If L has 0, then $f0 = 0$ implies $D(0, y) = 0$ for all $y \in L$

Proof: **(1)** Let $x, y \in L$.

Then $D(x,y) = D(x\wedge x, y) = [fx\wedge D(x,y)]\vee[fx\wedge D(x,y)] = fx\wedge D(x,y)$.

(2) Let $x,y,z\in L$. Then

$$\begin{aligned}[fx\vee fz]\wedge D(x\wedge z, y) &= [fx\vee fz]\wedge[[fz\wedge D(x,y)]\vee[fx\wedge D(z,y)]]\\ &= [[fx\vee fz]\wedge fz\wedge D(x,y)]\vee[[fx\vee fz]\wedge fx\wedge D(z,y)]\\ &= [fz\wedge D(x,y)]\vee[fx\wedge D(z,y)] = D(x\wedge z, y).\end{aligned}$$

(3) Suppose L has 0 and $f0=0$. Then,
by (1) above, $D(0,y) = f0\wedge D(0,y) = 0\wedge D(0,y) = 0$.

Corollary 3.9: If d is the trace of a symmetric bi- f -derivation D, then $dx = fx\wedge dx$ for all $x\in L$.

Theorem 3.10: If d is the trace of a symmetric bi- f -derivation on an assosiative ADL L, then $d(x\wedge y) = (fy\wedge dx)\vee D(x,y)\vee(fx\wedge dy)$.

Proof: Let $x,y\in L$. Then

$$\begin{aligned}d(x\wedge y) &= D(x\wedge y, x\wedge y)\\ &= [fy\wedge D(x, x\wedge y)]\vee[fx\wedge D(y, x\wedge y)]\\ &= [fy\wedge[[fy\wedge D(x,x)]\vee[fx\wedge D(x,y)]]]\vee[fx\wedge[[fy\wedge D(y,x)]\vee[fx\wedge D(y,y)]]]\\ &= (fy\wedge dx)\vee D(x,y)\vee(fx\wedge dy).\end{aligned}$$

Corollary 3.11: If d is the trace of a symmetric bi- f -derivation on an ADL L, then $fy\wedge dx\le d(x\wedge y)$.

Proof: Let $x,y\in L$. Then

$$\begin{aligned}d(x\wedge y) &= D(x\wedge y, x\wedge y)\\ &= [fy\wedge D(x, x\wedge y)]\vee[fx\wedge D(y, x\wedge y)]\\ &= [fy\wedge[[fy\wedge D(x,x)]\vee[fx\wedge D(x,y)]]]\vee[fx\wedge D(y, x\wedge y)]\\ &= [(fy\wedge dx)\vee D(x,y)]\vee[fx\wedge D(y, x\wedge y)].\end{aligned}$$

Thus $fy\wedge dx\le(fy\wedge dx)\vee D(x,y)\le d(x\wedge y)$.

Theorem 3.12: Let m be a maximal element of L and d be the trace of a symmetric bi- f -derivation D on L such that fm is also a maximal element. Then the following are equivalent.

1. d is an isotone map on L
2. $dx = fx\wedge dm$ for all $x\in L$
3. $d(x\wedge y) = dx\wedge dy$ for all $x,y\in L$
4. $d(x\vee y) = dx\vee dy$ for all $x,y\in L$.

Proof: $(1)\Rightarrow(2)$: Let $x\in L$. By Corollary 3.11, $fx\wedge dm\le d(m\wedge x) = dx$.
On the other hand, since d is an isotone, $d(x\wedge m)\le dm$. Thus $fm\wedge dx\le d(x\wedge m)\le dm$.
Therefore, $dx = fx\wedge dx = fm\wedge fx\wedge dx = fx\wedge fm\wedge dx\le fx\wedge dm$. Hence $dx = fx\wedge dm$.

$(2)\Rightarrow(3)$: Let $x,y\in L$. Then $d(x\wedge y) = x\wedge y\wedge dm = x\wedge dm\wedge y\wedge dm = dx\wedge dy$.
Then $d(x\wedge y) = f(x\wedge y)\wedge dm = fx\wedge fy\wedge dm = fx\wedge dm\wedge fy\wedge dm = dx\wedge dy.$

$(2)\Rightarrow(4)$: Let $x,y\in L$. Then $d(x\vee y) = (x\vee y)\wedge dm = (x\wedge dm)\vee(y\wedge dm) = dx\vee dy$.
Then $d(x\vee y) = f(x\vee y)\wedge dm = (fx\vee fy)\wedge dm = (fx\wedge dm)\vee(fy\wedge dm) = dx\vee dy.$
$(3)\Rightarrow(1)$ *and* $(4)\Rightarrow(1)$ are trivial.

Lemma 3.13: Let D be a symmetric bi- f -derivation on L. Then the following hold:

1. If D is isotone, then $D(x, y) = fx \wedge D(x \vee z, y)$
2. If f is a join homomorphism, then $D(x, y) = [fx \wedge D(x \vee z, y)] \vee D(x, y)$
3. If f is increasing, then $D(x, y) = [fx \wedge D(x \vee z, y)] \vee D(x, y)$

Proof: Let $x, y, z \in L$.

(1) Suppose D is an isotone function on L.

Then $D(x, y) \leq D(x \vee z, y)$. Thus $D(x, y) \wedge fx \wedge D(x \vee z, y) = D(x, y)$.

Therefore $D(x, y) \leq fx \wedge D(x \vee z, y)$.

Now, $D(x, y) = D((x \vee z) \wedge x, y) = [fx \wedge D(x \vee z, y)] \vee [f(x \vee z) \wedge D(x, y)]$.

Thus $fx \wedge D(x \vee z, y) \leq D(x, y)$. Hence $D(x, y) = fx \wedge D(x \vee z, y)$.

(2) Let f be a join-homomorphism on L. Then

$$
\begin{aligned}
D(x, y) &= D((x \vee z) \wedge x, y) \\
&= [fx \wedge D(x \vee z, y)] \vee [f(x \vee z) \wedge D(x, y)] \\
&= [fx \wedge D(x \vee z, y)] \vee [fx \vee fz) \wedge D(x, y)] \\
&= [fx \wedge D(x \vee z, y)] \vee [[fx \wedge D(x, y)] \vee [fz \wedge D(x, y)]] \\
&= [fx \wedge D(x \vee z, y)] \vee [D(x, y) \vee [fz \wedge D(x, y)]] \\
&= [fx \wedge D(x \vee z, y)] \vee D(x, y).
\end{aligned}
$$

(3) Let f be an increasing function on L. Then $fx \leq f(x \vee z)$.

Now,

$$
\begin{aligned}
D(x, y) &= D((x \vee z) \wedge x, y) \\
&= [fx \wedge D(x \vee z, y)] \vee [f(x \vee z) \wedge D(x, y)] \\
&= [fx \wedge D(x \vee z, y)] \vee [f(x \vee z) \wedge fx \wedge D(x, y)] \\
&= [fx \wedge D(x \vee z, y)] \vee [fx \wedge D(x, y)] \\
&= [fx \wedge D(x \vee z, y)] \vee D(x, y).
\end{aligned}
$$

Definition 3.14: Let D be a symmetric bi- f -derivation on L and $a \in L$. We define $F_a(L) = \{x \in L / D(a, x) \wedge fx = fx\}$.

Lemma 3.15: Let D be a symmetric bi- f -derivation on L where f is an increasing function and $a \in L$. Then $F_a(L)$ is an initial segment in L.

Proof: Let $x, y \in L$ with $x \leq y$ and $y \in Fix_a(L)$. Since f is an increasing function, $fx \leq fy$.

Now,

$$
\begin{aligned}
D(x, a) \wedge fx &= D(x \wedge y, a) \wedge fx \\
&= [[fy \wedge D(x, a)] \vee [fx \wedge D(y, a)]] \wedge fx \\
&= [[fy \wedge fx \wedge D(x, a)] \vee [fx \wedge fy \wedge D(y, a)]] \wedge fx \\
&= [[fx \wedge D(x, a)] \vee [fx \wedge D(y, a) \wedge fy]] \wedge fx \\
&= [D(x, a) \vee [fx \wedge fy]] \wedge fx \\
&= [D(x, a) \vee fx] \wedge fx \\
&= fx.
\end{aligned}
$$

Lemma 3.16: Let D be a join preserving symmetric bi- f -derivation on L where f is a join-homomorphism and $a \in L$. Then $x \vee y \in F_a(L)$ for all $x, y \in F_a(L)$.

Proof: Let $x, y \in F_a(L)$. Then

$$
\begin{aligned}
D(x\vee y,a)\wedge f(x\vee y) &= [D(x,a)\vee D(y,a)]\wedge f(x\vee y)\wedge f(x\vee y)\\
&= [[D(x,a)\vee D(y,a)]\wedge[fx\vee fy]]\wedge f(x\vee y)\\
&= [[D(x,a)\wedge[fx\vee fy]]\vee[D(y,a)]\wedge[fx\vee fy]]]\wedge f(x\vee y)\\
&= [[fx\vee[D(x,a)\wedge fy]]\vee[[D(y,a)\wedge fx]\vee fy]]\wedge f(x\vee y)\\
&= [[[fx\vee D(x,a)]\wedge[fx\vee fy]]\vee[[D(y,a)\vee fy]\wedge[fx\vee fy]]]\wedge f(x\vee y)\\
&= [fx\vee fy]\wedge f(x\vee y)\\
&= f(x\vee y)\wedge f(x\vee y)\\
&= f(x\vee y).
\end{aligned}
$$

Hence $x \vee y \in F_a(L)$.

Finally we conclude this paper with the following theorem, which is a direct consequence of Lemma 3.15 and Lemma 3.16.

Theorem 3.17: Let L be an ADL with 0 and D be a join preserving symmetric bi- f -derivation on L where f is a join-homomorphism and $a \in L$. Then $F_a(L)$ is a weak ideal of L.

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