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RELATION BETWEEN TOTAL DOMINATION NUMBER, ENERGY OF A GRAPH AND RANK

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ABSTRACT

Domination theory and energy of a graph are the fastest growing areas within graph theory. The energy of a graph is defined as the sum of the absolute values of all eigen values of its adjacency matrix of a graph. In this paper we present some sharp lower bounds which relate total domination number of a graph G, energy of G and rank of the incident matrix of some class of graphs.

Key Words: *Incidence matrix, total domination number, Domination number, Energy of graph and Rank of incidence matrix.*

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INTRODUCTION

C. Berge introduced the theory of domination. The concept of domination is originated from the game of chess played in ancient India. In 1862, C.F De Jaenisch considered the queens that can be placed on chess board such that every square is occupied by a queen or else guarded by at least one queen. The study of domination in graphs was further developed in late 1950's and 1960's beginning with C. Berge. [1] in 1958. Oysten Ore [2] introduced the terms dominating set and domination number. Recently in 1977, S.T. Hedetnieme and E. J. Cockayne introduced a new parameter called total domination number [3].

A set $D \subseteq V$ is a total dominating set if every vertex $v \in V$, there exists $u \in D, u \neq v$, such that u is adjacent to v or a subset D of V is called total dominating set in G if induced sub graph $\langle D \rangle$ has no isolated vertices i.e. $N(D) = V$. The smallest cardinality taken over all total dominating sets is called total domination number denoted by $\gamma_t(G)$ [4].

Eigen values and Eigen vectors provide insight into the geometry of the associated linear transformation. Energy of a graph is originated from theoretical chemistry. Chemists have shown that the experimental hears of formation of conjugated hydrocarbons are closely related to the total π-electron energy and the calculation of the total energy of all π- electrons in a conjugated hydrocarbon can be reduced to $E(G) = |\lambda_1(G)| + |\lambda_2(G)| + \cdots + |\lambda_n(G)|$ where $\lambda_i(G)$ are the Eigen values of corresponding graph [8]. The number of non zero rows in a row reduced form of a matrix A is called the rank of A denoted by $\rho(A)$. Rank of the matrix is the number of linearly independent rows or the number of linearly independent columns. A matrix always represents a linear transformation between two vector spaces. From the rank of the matrix we come to know several proposition about this linear transformation. Rank of the matrix equals the dimension of the linear manifold spanned by vertices $x_1, x_2 \ldots\ldots x_n$.

In this paper we find some sharp lower bounds which relate domination number of G, energy of G and rank of the incidence matrix of a graph. This is the paper motivated from the paper [11].

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Theorem 1: Let *G* be a complete graph without loops and multiple edges, $E(G)$ is the energy of graph *G*. $I(G)$ is the incidence matrix of graph G. $\rho(G) = Rank\ I(G)$. $\gamma_t(G)$ is total domination number of G, then

$$\gamma_t(G) = \left\lceil \frac{E(G)}{Rank\ I(G)} \right\rceil$$

Proof: Let *G* be a complete graph. The proof can be done in two ways.

i) Direct method and *ii*)Mathematical Induction

***i*) Direct method:**

Table-1: Complete graph

G	$\gamma_t(G)$	$E(G)$	$\rho(G)$	$\Delta(G)$	Eigen Values
K_2	2	2	2	-1	-1, 1
K_3	2	4	3	2	-1, -1, 2
K_4	2	6	4	-3	-1, -1, -1, 3
K_5	2	8	5	4	-1, -1, -1, -1, 4
K_6	2	10	6	-5	-1, -1, -1, -1, -1, 5
K_7	2	12	7	6	-1, -1, -1, -1, -1, -1, 6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
K_n	2	2(*n*-1)	*n*	$(-1)^n n$	$(n-1)1's$ & $(n-1)$

We know that $\gamma_t(G) = 2$ where *G* is a complete graph. $E(K_n) = 2(n-1)$

From the table we get,

$$\frac{E(G)}{Rank\ I(G)} = \frac{2(n-1)}{n} < \frac{2n}{n} < 2$$

$$\Rightarrow \left\lceil \frac{E(G)}{Rank\ \ I(G)} \right\rceil = \left\lceil \frac{2(n-1)}{n} \right\rceil = 2 = \gamma_t(G)$$ G- complete graph

***ii*) Mathematical induction:**

To prove that $\gamma_t(G) = \left\lceil \frac{E(G)}{Rank\ \ I(G)} \right\rceil$

For, K_3, $\gamma_t(G) = \left\lceil \frac{4}{3} \right\rceil = 2$

For, K_4, $\gamma_t(G) = \left\lceil \frac{6}{4} \right\rceil = 2$

Let the result is true for *n* = *k* i.e. $\gamma_t(G_k) = \left\lceil \frac{E(G_k)}{Rank\ \ I\ (G_k)} \right\rceil$

To prove the result true for *n* = *k*+1

i.e. $\gamma_t(G_{k+1}) = \left\lceil \frac{E(G_{k+1})}{Rank\ \ I\ (G_{k+1})} \right\rceil$

We know that
$E(G) \le E(G_{k+1}),\ Rank\ I(G_k) \le Rank\ I(G_{k+1})$

By inspection $2Rank\ I(G_{k+1}) > E(G_{k+1})$

Hence we conclude that LHS = RHS for *n* = *k*+1

Only for K_2,
$\gamma_t(G) = \frac{E(G)}{Rank\ \ I(G)}$ and $\gamma_t(G) = 2\gamma(G)$ for $k \ge 2$

Theorem 2: Let *P* be connected path with no loops and multiple edges, then

$$\gamma_t(P_n) \ge \left\lceil \frac{E(P_n)}{Rank\ \ I(P_n)} \right\rceil + \lfloor E(P_n) - RankI(P_n) \rfloor.$$

Proof: The results for few paths from the table 2.

For $n = 12$, $6 \geq \left\lceil \frac{14.529}{12} \right\rceil + \lfloor 14.529 - 12 \rfloor = 4$

For $n = 15$, $7 \geq \left\lceil \frac{15.752}{12} \right\rceil + \lfloor 15.752 - 12 \rfloor = 5$

For $n = 14$, $8 \geq \left\lceil \frac{17.132}{14} \right\rceil + \lfloor 17.132 - 14 \rfloor = 5$

For $n = 15$, $8 \geq \left\lceil \frac{18.306}{14} \right\rceil + \lfloor 18.306 - 14 \rfloor = 6$

To prove the result in general, we consider a complete graph.

We delete all the extra edges from a complete graph with n vertices in order to get a path P_n.

We write the following equations

$$E(P_n) < E(K_n),\ Rank\ I(P_n) \leq Rank\ I(K_n)$$

Therefore from the equation $\frac{E(P_n)}{Rank\ I(P_n)} \leq \left\lceil \frac{E(K_n)}{Rank\ I(K_n)} \right\rceil$

$$\Rightarrow \left\lceil \frac{E(K_n)}{Rank\ I(K_n)} \right\rceil = \frac{E(P_n)}{Rank\ I(P_n)} + k$$

Where *k* is a constant, which is chosen as $K = \lfloor E(P_n) - Rank\ I(P_n) \rfloor$

Therefore $\left\lfloor \frac{E(K_n)}{Rank\ I(K_n)} \right\rfloor \leq \left\lceil \frac{E(P_n)}{Rank\ I(P_n)} \right\rceil + \lfloor E(P_n) - Rank\ I(P_n) \rfloor$

But, $\gamma(P_n) \leq \left\lceil \frac{E(P_n)}{Rank\ I(P_n)} \right\rceil + \lfloor E(P_n) - Rank\ I(P_n) \rfloor \leq \gamma_t(P_n)$

$\Rightarrow \quad \gamma_t(P_n) \geq \left\lceil \frac{E(P_n)}{Rank\ I(P_n)} \right\rceil + \lfloor E(P_n) - Rank\ I(P_n) \rfloor$

Hence the theorem.

Table-2: PATH

G	$\gamma_t(G)$	$E(G)$	$\rho(G)$	$\Delta(G)$	Eigen Values
P_2	2	2	2	-1	± 1
P_3	2	2.828	2	0	$\pm 1.414, 0$
P_4	2	4.472	4	1	$\pm 1.618, \pm 0.618$
P_5	3	5.464	4	0	$\pm 1.732, \pm 1{,}0$
P_6	4	6.988	6	-1	$\pm 1.802, \pm 1.247, \pm 0.445$
P_7	4	8.054	6	0	$\pm 1.848, \pm 1.414{,}0.0765, \pm 0$
P_8	4	9.516	8	1	$\pm 1.879, \pm 1.532, \pm 1, \pm 0.347$
P_9	5	10.628	8	0	$\pm 1.902, \pm 1.618, \pm 1.176, \pm 0.618{,}0$
P_{10}	6	12.056	10	-1	$\pm 1.919, \pm 1.683, \pm 1.310, \pm 0.831, \pm 0.285$
P_{11}	6	13.192	10	0	$\pm 1.932, \pm 1.732, \pm 1.414, \pm 1, \pm 0.518{,}0$
P_{12}	6	14.529	12	1	$\pm 1.942, \pm 1.771, \pm 1.497, \pm 1.136, \pm 0.709, \pm 0.241$
P_{13}	7	15.752	12	0	$\pm 1.350, \pm 1.802, \pm 1.564, \pm 1.247, \pm 0.868, \pm 0.445{,}0$
P_{14}	8	17.132	14	-1	$\pm 1.956, \pm 1.827, \pm 1.618, \pm 1.338, \pm 1, \pm 0.618, \pm 0.209$
P_{15}	8	18.306	14	0	$\pm 1.962, \pm 1.848, \pm 1.663, \pm 1.414, \pm 1.111, \pm 0.765, \pm 0.390{,}0$
P_{16}	-	-	-	-	-----
P_{17}	-	-	-	-	-----
P_{18}	-	-	-	-	-----

Corollary 3: If $\rho(G)$ and $\gamma_t(G)$ are the rank and total domination numbers of path on *k* vertices then,
$\rho(P_{2k}) = 2k$

$\gamma_t(P_{4k}) = 2k$

$\rho(P_{4k}) = 2\gamma_t(P_{4k})$ otherwise $\rho(P_k) \geq \gamma_t(P_k)$

Theorem 4: Let a cycle C_n be a connected graph with no loops and multiple edges
Then, $\gamma_t(C_n) \geq \left\lceil \frac{E(C_n)}{Rank\ \ I(C_n)} \right\rceil + \lfloor E(C_n) - Rank\ I(C_n) \rfloor$

Proof: We can prove the above theorem similar to theorem 2

The inequality fails only for $C_4 and\ C_8$

Table-3: CYCLE

G	$\gamma_t(G)$	$E(G)$	$\rho(G)$	$\Delta(G)$	Eigen Values
C_3	2	4	3	2	-1,-1,2
C_4	2	4	2	0	-2,2,0,0
C_5	3	6.472	5	2	1.618,1.618,±0.618,2
C_6	4	8	6	-4	±2,±1,±1
C_7	4	8.988	7	2	-1.802,-1.802,-0.445,-0.445,±1.2472
C_8	4	9.656	6	0	±2,±1.414,±1.414,0,0
C_9	5	11.516	9	2	-1.879,-1.879,-1,0.347,0.347,2,1.532,1.532,-1
C_{10}	6	12.944	10	-4	±2,±1.616,±1.618,±0.618,±0.618
C_{11}	6	14.206	11	2	-1.919,-1.919,-1.310,-1.310,-1.310,-0.285,-0.285,0.831,0.831,2,1.683
C_{12}	6	14.928	10	0	±2,±1,±1,±1.732,±1.732,0,0
C_{13}	7	16.562	13	2	-1.942,-1.914,-1.497,-1.497,-0.709,-0.709,0.241,0.241,1.136,1.136,2,1.771,1.771
C_{14}	8	17.976	14	-4	±2,±1.802,±1.802,±1.247,±1.247,±0.445,±0.445
C_{15}	8	19.132	15	2	-1.956,-1.618,-1.618,-1,-1,-0.209,-0.209,0.618,0.618,1.338,1.338,2,1.827,1.827,-1.956
C_{16}	8	20.1094	14	0	-2,-1.847,-1.847,-1.414,-1.414,-0.765,-0.765,0,-0,0.765,0.765,1.414,1.414,1.847,1.847,2
C_{17}	-	-	-	-	------
C_{18}	-	-	-	-	------

Corollary 5: If $\rho(G)$ and $\gamma_t(G)$ are the rank and total domination numbers of cycle on *k* vertices then,
$\rho(C_{4k}) = 4k - 2$

$\gamma_t(C_{4k}) = 2k$

$\rho(C_k) = k$ except for $k = 4n, n = 1,2,3, \ldots$

$\gamma_t(C_k) \ \leq \rho(C_k)$

CONCLUSION AND SCOPE

We established the relation between Total domination number of a graph *G*, Energy of graph *G* and Rank of incidence matrix of some families of graphs.

The relation between these parameters can be extended to other classes of graphs and other types of domination.

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