

RELATION BETWEEN TOTAL DOMINATION NUMBER, ENERGY OF A GRAPH AND RANK

MANJULA C. GUDGERI*¹, SHAILAJA S. SHIRKOL², VARSHA³, PALLAVI I. KALMATH⁴

^{1,4}KLE Dr M S Sheshgiri college of Engineering and Technology,
Belagavi, Karnataka, 590008, INDIA.

²S. D. M College of engineering and Technology, Dharwar, Karnataka, INDIA.

³Angadi Institute of Technology, Belagavi, Karnataka, INDIA.

(Received On: 11-07-16; Revised & Accepted On: 29-07-16)

ABSTRACT

Domination theory and energy of a graph are the fastest growing areas within graph theory. The energy of a graph is defined as the sum of the absolute values of all eigen values of its adjacency matrix of a graph. In this paper we present some sharp lower bounds which relate total domination number of a graph G , energy of G and rank of the incidence matrix of some class of graphs.

Key Words: Incidence matrix, total domination number, Domination number, Energy of graph and Rank of incidence matrix.

AMS Subject Classification: 05C69, 05C50.

INTRODUCTION

C. Berge introduced the theory of domination. The concept of domination is originated from the game of chess played in ancient India. In 1862, C.F De Jaenisch considered the queens that can be placed on chess board such that every square is occupied by a queen or else guarded by at least one queen. The study of domination in graphs was further developed in late 1950's and 1960's beginning with C. Berge. [1] in 1958. Oysten Ore [2] introduced the terms dominating set and domination number. Recently in 1977, S.T. Hedetnieme and E. J. Cockayne introduced a new parameter called total domination number [3].

A set $D \subseteq V$ is a total dominating set if every vertex $v \in V$, there exists $u \in D, u \neq v$, such that u is adjacent to v or a subset D of V is called total dominating set in G if induced sub graph $\langle D \rangle$ has no isolated vertices i.e. $N(D) = V$. The smallest cardinality taken over all total dominating sets is called total domination number denoted by $\gamma_t(G)$ [4].

Eigen values and Eigen vectors provide insight into the geometry of the associated linear transformation. Energy of a graph is originated from theoretical chemistry. Chemists have shown that the experimental heats of formation of conjugated hydrocarbons are closely related to the total π -electron energy and the calculation of the total energy of all π - electrons in a conjugated hydrocarbon can be reduced to $E(G) = |\lambda_1(G)| + |\lambda_2(G)| + \dots + |\lambda_n(G)|$ where $\lambda_i(G)$ are the Eigen values of corresponding graph [8]. The number of non zero rows in a row reduced form of a matrix A is called the rank of A denoted by $\rho(A)$. Rank of the matrix is the number of linearly independent rows or the number of linearly independent columns. A matrix always represents a linear transformation between two vector spaces. From the rank of the matrix we come to know several proposition about this linear transformation. Rank of the matrix equals the dimension of the linear manifold spanned by vertices x_1, x_2, \dots, x_n .

In this paper we find some sharp lower bounds which relate domination number of G , energy of G and rank of the incidence matrix of a graph. This is the paper motivated from the paper [11].

Corresponding Author: Manjula C. Gudgeri*¹

**^{1,4}KLE Dr M S Sheshgiri college of Engineering and Technology,
Belagavi, Karnataka, 590008, INDIA.**

Theorem 1: Let G be a complete graph without loops and multiple edges, $E(G)$ is the energy of graph G . $I(G)$ is the incidence matrix of graph G . $\rho(G) = \text{Rank } I(G)$. $\gamma_t(G)$ is total domination number of G , then

$$\gamma_t(G) = \left\lceil \frac{E(G)}{\text{Rank } I(G)} \right\rceil$$

Proof: Let G be a complete graph. The proof can be done in two ways.

i) Direct method and ii) Mathematical Induction

i) Direct method:

Table-1: Complete graph

| G | $\gamma_t(G)$ | $E(G)$ | $\rho(G)$ | $\Delta(G)$ | Eigen Values |
|----------|---------------|----------|-----------|-------------|---------------------------|
| K_2 | 2 | 2 | 2 | -1 | -1, 1 |
| K_3 | 2 | 4 | 3 | 2 | -1, -1, 2 |
| K_4 | 2 | 6 | 4 | -3 | -1, -1, -1, 3 |
| K_5 | 2 | 8 | 5 | 4 | -1, -1, -1, -1, 4 |
| K_6 | 2 | 10 | 6 | -5 | -1, -1, -1, -1, -1, 5 |
| K_7 | 2 | 12 | 7 | 6 | -1, -1, -1, -1, -1, -1, 6 |
| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |
| K_n | 2 | $2(n-1)$ | n | $(-1)^n n$ | $(n-1)1$'s & $(n-1)$ |

We know that $\gamma_t(G) = 2$ where G is a complete graph. $E(K_n) = 2(n-1)$

From the table we get,

$$\frac{E(G)}{\text{Rank } I(G)} = \frac{2(n-1)}{n} < \frac{2n}{n} < 2$$

$$\Rightarrow \left\lceil \frac{E(G)}{\text{Rank } I(G)} \right\rceil = \left\lceil \frac{2(n-1)}{n} \right\rceil = 2 = \gamma_t(G) \text{ G- complete graph}$$

ii) Mathematical induction:

To prove that $\gamma_t(G) = \left\lceil \frac{E(G)}{\text{Rank } I(G)} \right\rceil$

For, $K_3, \gamma_t(G) = \left\lceil \frac{4}{3} \right\rceil = 2$

For, $K_4, \gamma_t(G) = \left\lceil \frac{6}{4} \right\rceil = 2$

Let the result is true for $n = k$ i.e. $\gamma_t(G_k) = \left\lceil \frac{E(G_k)}{\text{Rank } I(G_k)} \right\rceil$

To prove the result true for $n = k+1$

i.e. $\gamma_t(G_{k+1}) = \left\lceil \frac{E(G_{k+1})}{\text{Rank } I(G_{k+1})} \right\rceil$

We know that

$$E(G) \leq E(G_{k+1}), \text{ Rank } I(G_k) \leq \text{Rank } I(G_{k+1})$$

By inspection $2\text{Rank } I(G_{k+1}) > E(G_{k+1})$

Hence we conclude that LHS = RHS for $n = k+1$

Only for $K_2,$

$$\gamma_t(G) = \frac{E(G)}{\text{Rank } I(G)} \text{ and } \gamma_t(G) = 2\gamma(G) \text{ for } k \geq 2$$

Theorem 2: Let P be connected path with no loops and multiple edges, then

$$\gamma_t(P_n) \geq \left\lceil \frac{E(P_n)}{\text{Rank } I(P_n)} \right\rceil + \lfloor E(P_n) - \text{Rank } I(P_n) \rfloor.$$

Proof: The results for few paths from the table 2.

$$\text{For } n = 12, 6 \geq \left\lfloor \frac{14.529}{12} \right\rfloor + [14.529 - 12] = 4$$

$$\text{For } n = 15, 7 \geq \left\lfloor \frac{15.752}{12} \right\rfloor + [15.752 - 12] = 5$$

$$\text{For } n = 14, 8 \geq \left\lfloor \frac{17.132}{14} \right\rfloor + [17.132 - 14] = 5$$

$$\text{For } n = 15, 8 \geq \left\lfloor \frac{18.306}{14} \right\rfloor + [18.306 - 14] = 6$$

To prove the result in general, we consider a complete graph.

We delete all the extra edges from a complete graph with n vertices in order to get a path P_n .

We write the following equations

$$E(P_n) < E(K_n), \text{ Rank } I(P_n) \leq \text{ Rank } I(K_n)$$

$$\begin{aligned} \text{Therefore from the equation } \frac{E(P_n)}{\text{Rank } I(P_n)} &\leq \left\lfloor \frac{E(K_n)}{\text{Rank } I(K_n)} \right\rfloor \\ &\Rightarrow \left\lfloor \frac{E(K_n)}{\text{Rank } I(K_n)} \right\rfloor = \frac{E(P_n)}{\text{Rank } I(P_n)} + k \end{aligned}$$

Where k is a constant, which is chosen as $K = [E(P_n) - \text{Rank } I(P_n)]$

$$\text{Therefore } \left\lfloor \frac{E(K_n)}{\text{Rank } I(K_n)} \right\rfloor \leq \left\lfloor \frac{E(P_n)}{\text{Rank } I(P_n)} \right\rfloor + [E(P_n) - \text{Rank } I(P_n)]$$

$$\text{But, } \gamma(P_n) \leq \left\lfloor \frac{E(P_n)}{\text{Rank } I(P_n)} \right\rfloor + [E(P_n) - \text{Rank } I(P_n)] \leq \gamma_t(P_n)$$

$$\Rightarrow \gamma_t(P_n) \geq \left\lfloor \frac{E(P_n)}{\text{Rank } I(P_n)} \right\rfloor + [E(P_n) - \text{Rank } I(P_n)]$$

Hence the theorem.

Table-2: PATH

| G | $\gamma_t(G)$ | $E(G)$ | $\rho(G)$ | $\Delta(G)$ | Eigen Values |
|----------|---------------|--------|-----------|-------------|--|
| P_2 | 2 | 2 | 2 | -1 | ± 1 |
| P_3 | 2 | 2.828 | 2 | 0 | $\pm 1.414, 0$ |
| P_4 | 2 | 4.472 | 4 | 1 | $\pm 1.618, \pm 0.618$ |
| P_5 | 3 | 5.464 | 4 | 0 | $\pm 1.732, \pm 1, 0$ |
| P_6 | 4 | 6.988 | 6 | -1 | $\pm 1.802, \pm 1.247, \pm 0.445$ |
| P_7 | 4 | 8.054 | 6 | 0 | $\pm 1.848, \pm 1.414, 0.0765, \pm 0$ |
| P_8 | 4 | 9.516 | 8 | 1 | $\pm 1.879, \pm 1.532, \pm 1, \pm 0.347$ |
| P_9 | 5 | 10.628 | 8 | 0 | $\pm 1.902, \pm 1.618, \pm 1.176, \pm 0.618, 0$ |
| P_{10} | 6 | 12.056 | 10 | -1 | $\pm 1.919, \pm 1.683, \pm 1.310, \pm 0.831, \pm 0.285$ |
| P_{11} | 6 | 13.192 | 10 | 0 | $\pm 1.932, \pm 1.732, \pm 1.414, \pm 1, \pm 0.518, 0$ |
| P_{12} | 6 | 14.529 | 12 | 1 | $\pm 1.942, \pm 1.771, \pm 1.497, \pm 1.136, \pm 0.709, \pm 0.241$ |
| P_{13} | 7 | 15.752 | 12 | 0 | $\pm 1.350, \pm 1.802, \pm 1.564, \pm 1.247, \pm 0.868, \pm 0.445, 0$ |
| P_{14} | 8 | 17.132 | 14 | -1 | $\pm 1.956, \pm 1.827, \pm 1.618, \pm 1.338, \pm 1, \pm 0.618, \pm 0.209$ |
| P_{15} | 8 | 18.306 | 14 | 0 | $\pm 1.962, \pm 1.848, \pm 1.663, \pm 1.414, \pm 1.111, \pm 0.765, \pm 0.390, 0$ |
| P_{16} | - | - | - | - | ----- |
| P_{17} | - | - | - | - | ----- |
| P_{18} | - | - | - | - | ----- |

Corollary 3: If $\rho(G)$ and $\gamma_t(G)$ are the rank and total domination numbers of path on k vertices then,
 $\rho(P_{2k}) = 2k$

$$\gamma_t(P_{4k}) = 2k$$

$$\rho(P_{4k}) = 2\gamma_t(P_{4k}) \text{ otherwise } \rho(P_k) \geq \gamma_t(P_k)$$

Theorem 4: Let a cycle C_n be a connected graph with no loops and multiple edges

Then, $\gamma_t(C_n) \geq \left\lceil \frac{E(C_n)}{\text{Rank } I(C_n)} \right\rceil + [E(C_n) - \text{Rank } I(C_n)]$

Proof: We can prove the above theorem similar to theorem 2

The inequality fails only for C_4 and C_8

Table-3: CYCLE

| G | $\gamma_t(G)$ | $E(G)$ | $\rho(G)$ | $\Delta(G)$ | Eigen Values |
|----------|---------------|---------|-----------|-------------|---|
| C_3 | 2 | 4 | 3 | 2 | -1,-1,2 |
| C_4 | 2 | 4 | 2 | 0 | -2,2,0,0 |
| C_5 | 3 | 6.472 | 5 | 2 | 1.618,1.618, ± 0.618 ,2 |
| C_6 | 4 | 8 | 6 | -4 | $\pm 2, \pm 1, \pm 1$ |
| C_7 | 4 | 8.988 | 7 | 2 | -1.802,-1.802,-0.445,-0.445, ± 1.2472 |
| C_8 | 4 | 9.656 | 6 | 0 | $\pm 2, \pm 1.414, \pm 1.414, 0, 0$ |
| C_9 | 5 | 11.516 | 9 | 2 | -1.879,-1.879,-1,0.347,0.347,2,1.532,1.532,-1 |
| C_{10} | 6 | 12.944 | 10 | -4 | $\pm 2, \pm 1.616, \pm 1.618, \pm 0.618, \pm 0.618$ |
| C_{11} | 6 | 14.206 | 11 | 2 | -1.919,-1.919,-1.310,-1.310,-1.310,-0.285,-0.285,0.831,0.831,2,1.683 |
| C_{12} | 6 | 14.928 | 10 | 0 | $\pm 2, \pm 1, \pm 1, \pm 1.732, \pm 1.732, 0, 0$ |
| C_{13} | 7 | 16.562 | 13 | 2 | -1.942,-1.914,-1.497,-1.497,-0.709,-0.709,0.241,0.241,1.136,1.136,2,1.771,1.771 |
| C_{14} | 8 | 17.976 | 14 | -4 | $\pm 2, \pm 1.802, \pm 1.802, \pm 1.247, \pm 1.247, \pm 0.445, \pm 0.445$ |
| C_{15} | 8 | 19.132 | 15 | 2 | -1.956,-1.618,-1.618,-1,-1,-0.209,-0.209,0.618,0.618,1.338,1.338,2,1.827,1.827,-1.956 |
| C_{16} | 8 | 20.1094 | 14 | 0 | -2,-1.847,-1.847,-1.414,-1.414,-0.765,-0.765,0,-0,0.765,0.765,1.414,1.414,1.847,1.847,2 |
| C_{17} | - | - | - | - | ----- |
| C_{18} | - | - | - | - | ----- |

Corollary 5: If $\rho(G)$ and $\gamma_t(G)$ are the rank and total domination numbers of cycle on k vertices then,

$$\rho(C_{4k}) = 4k - 2$$

$$\gamma_t(C_{4k}) = 2k$$

$$\rho(C_k) = k \text{ except for } k = 4n, n = 1, 2, 3, \dots$$

$$\gamma_t(C_k) \leq \rho(C_k)$$

CONCLUSION AND SCOPE

We established the relation between Total domination number of a graph G , Energy of graph G and Rank of incidence matrix of some families of graphs.

The relation between these parameters can be extended to other classes of graphs and other types of domination.

REFERENCES

1. C. A. Coulson, "On the calculation of the energy in unsaturated hydrocarbon molecules", Proc. Cambridge, Phil. Soc.36 (1940), 201 – 203.
2. C.Berge, Theory of Graphs and its Applications, Methuen, London (1962).
3. Clemens Brand, Norbert Seifert, "Eigenvalues and Domination in Graphs", Mathematica Slovaca, Vol 46, 1996, Number 1, 33-39.
4. E.J.Cockayne, R.M.Dawes and S.T.Hedetniemi, Total Domination in Graphs, Newyork 10, (1980), pp 211-219.
5. Georgi. E Shilov. Linear Algebra. Dover Publications, Inc. Chapter 3.
6. I. Gutman, "The energy of graph", Ber. Math. Statist.Sekt. Forschungszentrum Graz. 103 (1978), 1-22.
7. I. Gutman, O.E.Polansky, Mathematical concepts in organic chemistry, Springer 1986
8. I. Gutman. "The energy of graph; old and new results", in; Abetten, A. Kohnert, R. Laue, A. Wassermann (Eds), Algebraic Combinatorics and Applications, Springer. (2000, 196-211).

9. I. Gutman, Topology and stability of conjugated hydrocarbons. The dependence of total π - electron energy on molecular topology – J. Serb. Chem. Soc, 70 (2005), 441-456.
10. M kamal kumar, Relation between Domination Number, Energy of Graph and Rank. Int.journal of Mathematics and Scientific computing Vol,1, No.1, 2011
11. O.Ore, Theory of Graphs, Amer, Math, Soc Colloq, Publ, 38 (1962)
12. S.T.Hedetniemi, P J.Slater, T.W.Haynes Fundamentals of Domination in Graphs, Marcel Dekker Newyork (1997)

Source of support: Nil, Conflict of interest: None Declared

[Copy right © 2016. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]