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ON A CLASS OF P-VALENT FUNCTIONS WITH ALTERNATING TYPE

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ABSTRACT

In this paper, we introduce a new subclass which are analytic and p-valent with alternating coefficients. Some results like coefficient estimation, radius of convexity, closure theorem, extreme points, convolution and inclusion property of p-valent functions are investigated.

Keywords: *analytic function, p-valent function, radius of convexity, convolution property, inclusion property.*

1. INTRODUCTION

Let $\mathcal{A}$(p)denote the class of *f* normalized univalent functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} (-1)^{n+1} a_{n+p} z^{n+p}, \quad (P \in N=\{1,2,3,...\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc E = {z : z ∈ C;|z| <1}.

A function f(z)∈$\mathcal{A}$ (p) is said to in the class of $S_p^*(\alpha)$ p-valently starlike function of order α (0≤α<p) if it satisfies, for z ∈E, the condition

$$Re\left(\frac{z f'(z)}{f(z)}\right) > \alpha \tag{1.2}$$

Furthermore, a function f(z)∈$\mathcal{A}$ (p) is said to in the class $\mathcal{K}_p(\alpha)$ of p-valently convex function of order α (0≤α<p) if it satisfies, for z ∈E, the condition

$$Re\left(1+\frac{z f''(z)}{f'(z)}\right) > \alpha \tag{1.3}$$

H. Özlem G ü ney and S. sümmer Eker, [1], Yi-Hui Xu, Qing Yang and Jin-Lin Liu [2], K.S.Padmanabhan and Ganeshan[3] and S.L.Shukla and Dastrath[4] have studied the certain classes of analytic functions with negative coefficients. In this paper we introduce a new subclass $S^*(\alpha,\beta,\xi,\gamma)$ of $\mathcal{A}$(p) defined by (1.1) and also satisfying condition:

$$\left|\frac{z\frac{f'(z)}{f(z)}+p}{2\xi\left[z\frac{f'(z)}{f(z)}+\alpha\right]-\gamma\left[z\frac{f'(z)}{f(z)}+p\right]}\right| < \beta \tag{1.4}$$

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where $\left(|z|<1, p\in N, 0\le\alpha<1, 0\le\beta<1, \frac{1}{2}<\xi\le 1, \frac{1}{2}<\gamma\le 1\right).$

We obtain the results like coefficient estimation, radius of convexity, closure theorem, extreme points, convolution and inclusion property of analytic and p-valent functions alternating type.

2. COEFFICIENT ESTIMATION

Theorem 2.1: A function f(z)∈$\mathcal{A}$ (p) is in the class $S^*(\alpha,\beta,\xi,\gamma)$ if and only if

$$2p+\sum_{n=p}^{\infty}[(p+n+1)+\beta(p+\alpha+1)-\gamma(n+p+1)\le 2\xi\beta[p(\xi-\gamma)+\alpha\xi] \tag{2.1}$$

Proof: Assume that inequality (2.1) holds true and let |z|=1. We show that f(z)∈ $S^*(\alpha,\beta,\xi,\gamma)$. From (1.4),

$$\left|\frac{z\frac{f'(z)}{f(z)}+p}{2\xi\left[z\frac{f'(z)}{f(z)}+\alpha\right]-\gamma\left[z\frac{f'(z)}{f(z)}+p\right]}\right|$$

$$=\left|\frac{2p+\sum_{n=p}^{\infty}(-1)^{n+1}(p+n+1)a_{n+1}z^{n+1-p}}{2p[(\xi-\gamma)+2\xi\alpha]-\sum_{n=p}^{\infty}(-1)^{n+1}[2\xi(\alpha+n+1)-\gamma(p+n+1)]a_{n+1}z^{n+1-p}}\right|$$

$$\le\frac{2p+\sum_{n=p}^{\infty}(-1)^{n+1}(p+n+1)a_{n+1}}{2p[(\xi-\gamma)+2\xi\alpha]-\sum_{n=p}^{\infty}(-1)^{n+1}[2\xi(\alpha+n+1)-\gamma(p+n+1)]a_{n+1}} \tag{2.2}$$

Above inequality is bounded above by β if,

$$2p[(\xi-\gamma)+2\xi\alpha]-\sum_{n=p}^{\infty}(-1)^{n+1}[2\xi(\alpha+n+1)-\gamma(p+n+1)]a_{n+1}\le 2\beta[\xi(p+\alpha)-p\gamma]$$

Hence by maximum modulus theorem, we have f(z) ∈ $S^*(\alpha,\beta,\xi,\gamma)$.

To prove the converse, assume that

$$\left|\frac{z\frac{f'(z)}{f(z)}+p}{2\xi\left[z\frac{f'(z)}{f(z)}+\alpha\right]-\gamma\left[z\frac{f'(z)}{f(z)}+p\right]}\right|<\beta$$

$$\left|\frac{2p+\sum_{n=p}^{\infty}(-1)^{n+1}(p+n+1)a_{n+1}z^{n+1-p}}{2p[(\xi-\gamma)+2\xi\alpha]-\sum_{n=p}^{\infty}(-1)^{n+1}[2\xi(\alpha+n+1)-\gamma(p+n+1)]a_{n+1}z^{n+1-p}}\right|<\beta$$

Note that $\left|\mathrm{Re}(z)\right| \le |z|$ for all z, and so

$$\mathrm{Re}\left|\frac{2p+\sum_{n=p}^{\infty}(-1)^{n+1}(p+n+1)a_{n+1}z^{n+1-p}}{2p[(\xi-\gamma)+2\xi\alpha]-\sum_{n=p}^{\infty}(-1)^{n+1}[2\xi(\alpha+n+1)-\gamma(p+n+1)]a_{n+1}z^{n+1-p}}\right|<\beta \tag{2.3}$$

Choosing value of z on real axis so that $z\frac{f'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.3) and allowing $z \to 1$ through the real values we obviously obtained required assertion (2.1).

Corollary 2.1 A function $f(z) \in S^*(\alpha,\beta,\xi,\gamma)$ then

$$a_{n+1} \le \frac{2\xi\beta\,[p(\xi-\gamma)+\alpha\xi]}{(-1)^{n+1}[(p+n+1)+\beta\,(2\xi(\alpha+n+1)-\gamma(p+n+1))]}$$ for $n \in N_0$ with equality for f(z) given by,

$$f(z)=z^p+\frac{2\xi\beta\,[p(\xi-\gamma)+\alpha\xi]}{(-1)^{n+1}[(p+n+1)+\beta\,(2\xi(\alpha+n+1)-\gamma(p+n+1))]}z^{p+n},(n\in N_0) \tag{2.4}$$

Corollary 2.2: A function $f(z) \in S^*(\alpha,\beta,\xi,\gamma)$ and p = 1 then

$$a_{n+1} \le \frac{2\xi\beta[(\xi-\gamma)+\alpha\xi]}{(-1)^{n+1}[(n+2)+\beta\,(2\xi(\alpha+n+1)-\gamma(n+2))]}$$ for $n \in N_0$ with equality for f(z) given by,

$$f(z)=z+\frac{2\xi\beta[(\xi-\gamma)+\alpha\xi]}{(-1)^{n+1}[(n+2)+\beta\,(2\xi(\alpha+n+1)-\gamma(n+2))]}z^{1+n},(n\in N_0) \tag{2.5}$$

Corollary 2.3 A function $f(z) \in S^*(\alpha,\beta,1,1)$ then

$$a_{n+1} \le \frac{2\beta\alpha}{(-1)^{n+1}[(p+n+1)+\beta\,(2(\alpha+n+1)-\gamma(p+n+1))]}$$ for $n \in N_0$ with equality for f(z) given by,

$$f(z)=z^p+\frac{2\beta\alpha}{(-1)^{n+1}[(p+n+1)+\beta\,(2(\alpha+n+1)-\gamma(p+n+1))]}z^{p+n}\ ,(n\in N_0) \tag{2.6}$$

3. RADIUS OF CONVEXITY AND STARLIKENESS

Theorem 3.1: If $f(z) \in \mathcal{A}(p)$ is in the class $S^*(\alpha,\beta,\xi,\gamma)$ then f(z)p-valently convex in

$$0<|z|<R_1=\inf_n\left[\frac{p^2\left[2p+(-1)^{n+1}(p+n+1)+2\xi(p+n+1)-\gamma(p+n+1)\right]}{2\beta[\xi(p+\alpha)-p\gamma](n+1)^2}\right]^{\frac{1}{n+1-p}} \tag{3.1}$$

The estimate is sharp for

$$f(z)=z^p+\frac{2\xi\beta\,[p(\xi-\gamma)+\alpha\xi]}{(-1)^{n+1}[(p+n+1)+\beta\,\,(2\xi(\alpha+n+1)-\gamma(p+n+1)]}z^{p+n}\quad(n\in N_0) \tag{3.2}$$

Proof: It is sufficient to show that,

$$\left|\frac{1+z\frac{f''(z)}{f'(z)}+p}{1+z\frac{f''(z)}{f'(z)}-p}\right| \le 1 \quad \text{for } 0<|z|<R$$

$$\left|\frac{1+z\frac{f''(z)}{f'(z)}+p}{1+z\frac{f''(z)}{f'(z)}-p}\right| \le \left|\frac{2p^2 z^{p-1}+\sum_{n=1}^{\infty}(-1)^{n+1}(1+n+p)a_{n+1}z^n}{z^{p-1}+\sum_{n=1}^{\infty}(-1)^{n+1}(1+n-p)a_{n+1}z^n}\right|$$

$$\le \frac{2p^2+\sum_{n=1}^{\infty}(-1)^{n+1}(1+n+p)a_{n+1}|z|^{n-p+1}}{\sum_{n=1}^{\infty}(-1)^{n+1}(1+n-p)a_{n+1}|z|^{n-p+1}}$$

The last expression is bounded by 1 provided,

$$\sum_{n=1}^{\infty}\left(\frac{n+1}{p}\right)^2 a_{n+1}|z|^{n-p+1} \le 1 \tag{3.3}$$

Also from theorem 1, we have

$$\frac{2p+\sum_{n=p}^{\infty}(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]a_{n+1}}{2\beta[\xi(p+\alpha)-p\gamma]} \le 1$$

Thus (3.3) is satisfied if,

$$\sum_{n=1}^{\infty}\left(\frac{n+1}{p}\right)^2 a_{n+1}|z|^{n-p+1} \le \frac{2p+\sum_{n=p}^{\infty}(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]a_{n+1}}{2\beta[\xi(p+\alpha)-p\gamma]} \tag{3.4}$$

Solving for |z| we get,

$$|z|=\inf_n\left[\frac{p^2\left[2p+(-1)^{n+1}(p+n+1)+2\xi(p+n+1)-\gamma(p+n+1)\right]}{2\beta\,[\xi(p+\alpha)-p\gamma](n+1)^2}\right]^{\frac{1}{n+1-p}} \tag{3.5}$$

Substituting $|z|<R_1$ in (3.5) we obtained required assertion (3.1).

Corollary 3.1: A function f(z)$\in S^*(\alpha,\beta,\xi,1)$ then f(z)is convex in the disc

$$0<|z|<R_2=\inf_n\left[\frac{p^2\left[2p+(-1)^{n+1}(p+n+1)+2\xi(p+n+1)-(p+n+1)\right]}{2\beta[\xi(p+\alpha)-p](n+1)^2}\right]^{\frac{1}{n+1-p}} \tag{3.6}$$

The estimate is sharp for the function

$$f(z)=z^p+\frac{2\xi\beta[p(\xi-1)+\alpha\xi]}{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-(p+n+1)]}z^{p+n} \quad (n\in N_0) \tag{3.7}$$

Corollary 3.2: A function f(z)∈ $S^*(\alpha,\beta,1,1)$ then f(z)is convex in the disc

$$0<|z|<R_3=\inf_n\left[\frac{p^2\left[2p+(-1)^{n+1}(p+n+1)+2(p+n+1)-(p+n+1)\right]}{2\beta[(p+\alpha)-p](n+1)^2}\right]^{\frac{1}{n+1-p}} \tag{3.8}$$

The estimate is sharp for the function

$$f(z)=z^p+\frac{2\beta\alpha}{(-1)^{n+1}[(p+n+1)+\beta(2(\alpha+n+1)-(p+n+1)]}z^{p+n} \tag{3.9}$$

Corollary 3.3: A function f(z)∈ $S^*(\alpha,\beta,1,1)$ then f(z)is convex in the disc

$$0<|z|<R_4=\inf_n\left[\frac{p^2\left[2p+(-1)^{n+1}(p+n+1)+2(p+n+1)-(p+n+1)\right]}{2\beta[(p+\alpha)-p](n+1)^2}\right]^{\frac{1}{n+1-p}} \tag{3.10}$$

The estimate is sharp for the function

$$f(z)=z^p+\frac{2\beta\alpha}{(-1)^{n+1}[(p+n+1)+\beta(2(\alpha+n+1)-(p+n+1)]}z^{p+n} \tag{3.11}$$

Theorem 3.2: If f(z)∈$\mathcal{A}$ (p) is in the class $S^*(\alpha,\beta,\xi,\gamma)$ then f(z)p-valently convex in

$$0<|z|<R_5=\inf_n\left[\frac{p^2\left[2p+(-1)^{n+1}(p+n+1)+2\xi(p+n+1)-\gamma(p+n+1)\right]}{2\beta[\xi(p+\alpha)-p\gamma](n+1)^2}\right]^{\frac{1}{n+1-p}} \tag{3.12}$$

Proof: It is sufficient to show that ,

$$\left|\frac{z\frac{f''(z)}{f'(z)}+p}{z\frac{f''(z)}{f'(z)}-p}\right|\le 1 \quad\text{for}\quad 0<|z|<R_5$$

The rest of the details fairly straight forward and are thus omitted.

4. EXTREME POINTS

Theorem 4.1: If $f_{p-n}(z)=z^p$ and

$$f_{p+n}(z)=z^p+\frac{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1)]}z^{p+n}\quad (n\in N_0) \tag{4.1}$$

Then f(z)∈ $S^*(\alpha,\beta,\xi,\gamma)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=-1}^{\infty}\lambda_{p+n}f_{n+1}(z)\quad\text{and}\quad\sum_{n=-1}^{\infty}\lambda_{p+n}=1$$

Proof: Assume

$$f(z)=\sum_{n=-1}^{\infty}\lambda_{p+n}f_{n+1}(z)$$

Using equation (1.1),

$$f(z) = z^p + \sum_{n=p}^{\infty} \lambda_{1+n} \frac{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1)]} z^{1+n} \quad (n \in N_0) \tag{4.2}$$

Notice that,

$$\sum_{n=-1}^{\infty} \lambda_{1+n} = 1 - \lambda_{p-1} \le 1$$

Which implies that then f(z) $\in S^*(\alpha,\beta,\xi,\gamma)$.

Conversely, let f(z) $\in S^*(\alpha,\beta,\xi,\gamma)$. Then by corollary (2.1)

$$a_{n+1} \le \frac{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1)]}, \ (n \in N_0)$$

Setting,

$$\lambda_{n+1} \le \frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1)]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]} a_{1+n}, \ (n \in N_0)$$

and

$\lambda_{p-1} = 1 - \sum_{n=p}^{\infty} \lambda_{1+n} f_{1+n}(z)$. We obtained $f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{n+1}(z)$

We complete the proof of theorem.

5. CLOSURE THEOREM

Theorem 5.1: If

$$f_j(z) = z^p + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1,j} z^{n+1}, (a_{n+1} \ge 0, j = 1,2,3,...) \tag{5.1}$$

be in the class f(z) $\in S^*(\alpha,\beta,\xi,\gamma)$. Then the function $g(z) = \sum_{n=p}^{\infty} c_j f_j(z)$ also belongs to the class f(z) $\in S^*(\alpha,\beta,\xi,\gamma)$

if $\sum_{n=p}^{\infty} c_j = 1.$

Proof: Let

$$g(z) = \sum_{n=p}^{\infty} c_j z^p + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1,j} z^{n+1}$$

$$= Z^p + \sum_{n=p}^{\infty} \sum_{j=p}^{1} C_j (-1)^n a_{n+1,j} z^{n+1}$$

$$= z^p + \sum_{n=p}^{\infty} (-1)^{n+1} C_{n+1,j} z^{n+1}$$

Where $C_{n+1,j} = \sum_{j=1}^{1} C_j a_{n+1,j}$

Notice that $f \in S^{*}(\alpha,\beta,\xi,\gamma)$ since

$$\sum_{n=p}^{\infty}\frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}$$

$$=\sum_{j=1}^{\infty}C_j\sum_{j=p}^{\infty}\frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}a_{n+1,j}$$

$$\le\sum_{j=1}^{\infty}C_j=1 \text{ since } f_j(s)\in f\in S^{*}(\alpha,\beta,\xi,\gamma)$$

Theorem 5.2: Let $f_j = Z^p + \sum_{n=p}^{\infty} a_{n+1,j}Z^{n+1}, a_{n+1}\ge 0, j=1,2,3,...$ be in the class $S^{*}(\alpha,\beta,\xi,\gamma)$ .Then the function $h(z)=\frac{1}{m}\sum_{n=p}^{\infty}f_j(z)$ also belongs to the class $S^{*}(\alpha,\beta,\xi,\gamma)$.

Proof: We have,

$$h(z)=\frac{1}{m}\sum_{n=p}^{\infty}f_j(z)$$

$$h(z)=Z^p+\sum_{n=p}^{\infty}\frac{1}{m}\sum_{j=1}^{\infty}a_{n+1,j}z^{n+1}$$

$$=Z^p+\sum_{n=p}^{\infty}d_k z^k \quad \text{where } d_k=\frac{1}{m}\sum_{j=1}^{\infty}a_{n+1,j}$$

Since $f_j \in S^{*}(\alpha,\beta,\xi,\gamma)$ from theorem 1, we have

$$\sum_{n=p}^{\infty}\frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}a_{n+1,j}\le 1 \tag{5.2}$$

Now $h(z)\in S^{*}(\alpha,\beta,\xi,\gamma)$ since

$$\sum_{n=p}^{\infty}\frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}d_k$$

$$=\sum_{n=p}^{\infty}\frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}\frac{1}{m}\sum_{j=1}^{m}a_{n+1,j}$$

$$=\frac{1}{m}\sum_{j=1}^{m}\sum_{n=p}^{\infty}\frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}a_{n+1,j}$$

= 1 by theorem (5.1)

Therefore $f(z)\in S^{*}(\alpha,\beta,\xi,\gamma)$

6. CONVOLUTION AND INCLUSION PROPERTY

For

$$f(z)=z^p+\sum_{n=p}^{\infty}(-1)^{n+1}a_{n+1}z^{n+1}, a_{n+1}\ge 0$$

$$g(z)=z^p+\sum_{n=p}^{\infty}(-1)^{n+1}b_{n+1}z^{n+1}, b_{n+1}\ge 0$$

in $f(z) \in S^*(\alpha,\beta,\xi,\gamma)$ the convolution of $f(z)*g(z)$ is defined by,

$$f(z)*g(z) = z^p + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} b_{n+1} z^{n+1}, \; a_{n+1}b_{n+1} \ge 0$$

Theorem 6.1: Let f(z) and g(z) belongs to $S^*(\alpha,\beta,\xi,\gamma)$ the convolution of $f(z)*g(z) \in S^*(\alpha,\beta,\xi,\gamma)$ for

$$\eta \ge \frac{2\xi\beta[p(\xi-\gamma)+\alpha\xi](p+n+1)}{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]^2 - 2\xi\beta[p(\xi-\gamma)+\alpha](2\xi(\alpha+n+1)-\gamma(p+n+1))} \tag{6.1}$$

Proof: Since f(z) and g(z) belongs to $S^*(\alpha,\beta,\xi,\gamma)$ and so

$$\sum_{n=p}^{\infty} \frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]} a_{n+1} \le 1$$

and

$$\sum_{n=p}^{\infty} \frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]} b_{n+1} \le 1$$

We need to find small number η such that

$$\sum_{n=p}^{\infty} \frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]} a_{n+1}b_{n+1} \le 1$$

Using Cauchy Schwartz inequality; we have

$$\sum_{n=p}^{\infty} \frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]} \sqrt{a_{n+1}b_{n+1}} \le 1 \tag{6.2}$$

Thus it is enough to show that,

$$\sum_{n=p}^{\infty} \frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]} a_{n+1}b_{n+1}$$
$$\le \sum_{n=p}^{\infty} \frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]} \sqrt{a_{n+1}b_{n+1}}$$

That is

$$\sqrt{a_{n+1}b_{n+1}} \le \frac{\eta(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{\beta(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]} \tag{6.3}$$

$$\sqrt{a_{n+1}b_{n+1}} \le \frac{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]} \tag{6.4}$$

In view of (6.3) and (6.4) it is enough to show that,

$$\frac{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}$$
$$\le \frac{\eta(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{\beta(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}$$

On simplifying, we get

$$\eta \ge \frac{2\xi\beta[p(\xi-\gamma)+\alpha\xi](p+n+1)}{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]^2 - 2\xi\beta[p(\xi-\gamma)+\alpha](2\xi(\alpha+n+1)-\gamma(p+n+1))}$$

We complete the proof of theorem.

Next we state that another inclusion theorem for the class $S^*(\alpha,\beta,\xi,\gamma)$.

Theorem 6.2: Let $f(z), g(z) \in S^*(\alpha,\beta,\xi,\gamma)$ then

$$h(z) = z^p + \sum_{n=p}^{\infty} (a_{n+1}^2 + b_{n+1}^2) z^{n+1} \text{ in } S^*(\alpha,\beta,\xi,\gamma)$$

where

$$\delta \geq \frac{4\xi\beta[p(\xi-\gamma)+\alpha\xi](p+n+1)}{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]^2 - 4\xi\beta[p(\xi-\gamma)+\alpha](2\xi(\alpha+n+1)-\gamma(p+n+1))} \tag{6.5}$$

Proof: $f(z), g(z) \in S^*(\alpha,\beta,\xi,\gamma)$ and hence

$$\sum_{n=p}^{\infty}\left[\frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}\right]^2 a_{n+1}^2$$

$$\leq \sum_{n=p}^{\infty}\left[\frac{(-1)^{n+1}[(p+\eta+1)+\beta(2\xi(\alpha+\eta+1)-\gamma(p+\eta+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]} a_{n+1}\right]^2 \leq 1 \tag{6.6}$$

Similarly,

$$\sum_{n=p}^{\infty}\left[\frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]}\right]^2 b_{n+1}^2$$

$$\leq \sum_{n=p}^{\infty}\left[\frac{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\beta[p(\xi-\gamma)+\alpha\xi]} b_{n+1}\right]^2 \leq 1 \tag{6.7}$$

We have to show that,

$$\sum_{n=p}^{\infty}\left[\frac{(-1)^{n+1}[(p+n+1)+\delta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\delta[p(\xi-\gamma)+\alpha\xi]}\right](a_{n+1}^2 + b_{n+1}^2) \leq 1 \tag{6.8}$$

Adding (6.5) and (6.6), we get

$$\sum_{n=p}^{\infty}\left[\frac{(-1)^{n+1}[(p+n+1)+\delta(2\xi(\alpha+n+1)-\gamma(p+n+1))]}{2\xi\delta[p(\xi-\gamma)+\alpha\xi]}\right]^2 (a_{n+1}^2 + b_{n+1}^2) \leq 1 \tag{6.9}$$

it is enough to show that,

$$\left[\frac{(-1)^{n+1}[(p+n+1)+\delta(2\xi(\alpha+n+1)-\gamma(p+n+1))}{2\xi\delta[p(\xi-\gamma)+\alpha\xi]}\right]$$

$$\leq \frac{1}{2}\left[\frac{(-1)^{n+1}[(p+n+1)+\delta(2\xi(\alpha+n+1)-\gamma(p+n+1))}{2\xi\delta[p(\xi-\gamma)+\alpha\xi]}\right]^2$$

This implies that

$$\delta \geq \frac{4\xi\beta[p(\xi-\gamma)+\alpha\xi](p+n+1)}{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-\gamma(p+n+1))]^2 - 4\xi\beta\,[p(\xi-\gamma)+\alpha](2\xi(\alpha+n+1)-\gamma(p+n+1))}$$

We complete the proof of theorem.

In this paper, we derived interesting properties of subclass $S^*(\alpha,\beta,\xi,\gamma)$ which are analytic and p-valent. The results like coefficient estimation, radius of convexity, closure theorem, extreme points, convolution and inclusion property of subclass $S^*(\alpha,\beta,\xi,\gamma)$ are obtained.

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