International Journal of Mathematical Archive-7(6), 2016, 111-113
IJMA Available online through www.ijma.info ISSN 2229 – 5046

ON A THEROREM ABOUT GENERALIZED METRIC SPACES

K. SUJATHA[1], S. RAGAMAYI*[2], CHERUVU KRISHNAVENI[3]

[1]Department of Mathematics,
Vijaya Engineering College, Ammapalem, Khammam-507305, Andhra Pradesh, India.

[2]Department of Mathematics,
K L University, Vaddeswaram, Guntur-522502, Andhra Pradesh, India.

[3]Department of Mathematics,
Maris Stella College, Vijayawada-520008, Andhra Pradesh, India.

(Received On: 15-06-16; Revised & Accepted On: 30-06-16)

ABSTRACT

Proposition 2 of [1] is false. We give an example to show this and then prove a correct version of the same.

1991 Mathematics subject classification: *54H25; 47H10.*

Keywords and Phrases: *Semimetric spaces, generalized metric spaces.*

1. INTRODUCTION

We find the following result in [1].

Proposition 2: If (X, d) is a generalized metric space which satisfies Axiom III, then the distance function is continuous. The purpose of this paper is to give an example to show that the above result is false and prove a correct version of the theorem.

2. MAIN RESULT

Definition 2.1: Let X be a nonempty set. A mapping $d:X\times X \to R$ is called a generalized metric on X and (X, d) is called a generalized metric space (gms) if for all x, y $\in$ X and all points u, v $\in$ X distinct among themselves and each distinct from x and y,
(i) $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y, x) \geq 0$
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Definition 2.2: Axiom III of [1]: A gms (X, d) is said to satisfy axiom III if to each pair of points $x \neq y$ in X there corresponds $r_{x,y} > 0$ such that $r_{x,y} \leq d(x, z) + d(y, z)$ for all z in X.

In [1] we find the following proposition:

If (X, d) is a gms which satisfies Axiom III, then the distance function is continuous.

To see that the proposition is false, consider the following example:

Corresponding Author: S. Ragamayi[2], [2]Department of Mathematics,*
K L University, Vaddeswaram, Guntur-522502, Andhra Pradesh, India.

Example 2.3: Let X = $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}, \ldots\ldots\}$.

On X define $d(\frac{1}{m}, \frac{1}{n})=1$ for m, n ∈ N and $d(\frac{1}{m}, 0)= d(0, \frac{1}{m}) = \frac{1}{m}$ for m∈N.

If {u, v}, {x, y} are disjoint subsets of X with u≠v then the sum
d(x, u) + d(u, v) + d(v, y), d(u, v) =1 if x = 0, d(v, y) = 1 if u =0 and d(x, y) = 1 if x≠0≠y and hence d(x, u) + d(u, v) + d(v, y) ≥ 1 ≥ d(x, y). It is clear that (X, d) is a gms. It is also clear that a sequence converges to a nonzero element c of X if it has a tail $x_n, x_{n+1}, \ldots\ldots$all of whose terms are equal to c while any other convergent sequence converges to 0. The sequence $\{\frac{1}{n}\}$ converges to 0, but it is not Cauchy. In this gms, the sequence $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$ both converge to 0 but $\lim d(x_n, y_n) = 1 \neq d(0,0)$. Hence 'd' is not continuous. Since limits are unique in (X, d), it is clear that (X, d) satisfies Axiom III. However, the following modification of the proposition 2 of [1] is valid:

We make use of the following result from [1] in the following proof:

Proposition 1: In a semimetric space, Axiom III is equivalent to the assertion that limits are unique.

Proposition 2.4: Let (X, d) be a gms. Then d is continuous if and only if (X, d) satisfies Axiom III and Cauchy sequences in (X, d) are convergent.

Proof: Suppose d is continuous. If $\lim d(x_n, x) = 0 = d(x_n, y)$, by continuity of d, we have $d(x_n, x_n)\to d(x, y)$. But $d(x_n, x_n)\to 0$. Hence d(x, y) = 0 and x = y. This proves uniqueness of limits. Hence (X, d) satisfies Axiom III. We shall now prove that convergent sequences in (X, d) are Cauchy. Let $\{x_n\}$ converges to x in (X, d). If all but finitely many terms of $\{x_n\}$ are different from x, let $\{x_{n_k}\}$ be the subsequence of $\{x_n\}$ formed by terms different from x. It will be proved that $\{x_n\}$ is Cauchy, if we prove that $\{x_{n_k}\}$ is Cauchy. No term of $\{x_{n_k}\}$ can repeat infinitely many times since each $x_{n_k} \neq$ x and since limits are unique. Let $\{x_{n_{k_l}}\}$ be the subsequence formed by retaining the first occurrence of a term and by dropping all the subsequent repetitions of it. If we denote $x_{n_{k_l}}$ by y_l then the y_l's are pairwise distinct, different from x and $\lim y_l = x$. If we prove that $\{y_l\}$ is a Cauchy sequence then it will be established that $\{x_{n_k}\}$ and hence $\{x_n\}$ is Cauchy.

Now $d(y_l, y_m) \leq d(y_l, x) + d(x, y_{m+1}) + d(y_{m+1}, y_m)$.

Since $y_l \to x$, there exists a positive integer n_1 such that $d(y_l, x) < \frac{\epsilon}{3}$, $d(x, y_{m+1}) < \frac{\epsilon}{3}$ for l, m ≥ n_1. By the continuity of 'd', since $y_m \to x$ and $y_{m+1} \to x$, we have $\lim d(y_m, y_{m+1}) = 0$.

Hence there exists $n_2 \in N$ such that $d(y_m, y_{m+1}) < \frac{\epsilon}{3}$ if $n \geq n_2$. Hence $d(y_l, y_m) < \epsilon$ for l, m ≥ max{n_1, n_2}. This proves that $\{y_l\}$ is a Cauchy sequence.

To prove the converse, suppose (X, d) satisfies Axiom III and convergent sequences in (X, d) are Cauchy.

Suppose $x_n \to x$, $y_n \to y$. We wish to prove that $d(x_n, y_n) \to d(x, y)$.

Case (i): Suppose $\{x_n\}$ and $\{y_n\}$ are eventually constant.

Since (X, d) satisfies Axiom III, limits are unique and hence there exists n_0 such that $n \geq n_0$ implies $x_n = x$ and $y_n = y$ and, evidently, $d(x_n, y_n) \to d(x, y)$.

Case (ii): Suppose one of the sequences, say $\{x_n\}$, is eventually constant and the other $\{y_n\}$ is not eventually constant. There is no harm in assuming that $x_n = x$ for every n. we need to prove that $\lim d(x, y_n) = d(x, y)$. Since infinitely many y_n's are different from y, we can remove all occurrences of y from $\{y_n\}$ and still obtain an infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$. Since each y_{n_k} is different from y and since limits of sequences are unique, no term of $\{y_{n_k}\}$ can repeat infinitely often. In case of repetitions, let us retain the first occurrence and remove all the subsequent repetitions to obtain an infinite subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that no two terms of $\{y_{n_{k_j}}\}$ are equal. Proving that

$\lim d(x, y_{n_{k_j}}) = d(x, y)$ is as good as proving that $\lim d(x_n, y_n) = d(x, y)$.

Denote $y_{n_{k_j}}$ by z_j.

$$d(x, y) \leq d(x, z_j) + d(z_j, z_{j+1}) + d(z_{j+1}, y).$$

Taking limits as $j \to \infty$, we obtain $d(x, y) \leq \liminf d(x, z_j)$.

Also $d(x, z_j) \leq d(x, y) + d(y, z_{j+1}) + d(z_{j+1}, z_j)$.

Taking limits as $j \to \infty$, $\limsup d(x, z_j) \leq d(x, y)$.

In deriving both the above inequalities, we have used the fact that $\{z_j\}$ is Cauchy.

Hence $\lim d(x, z_j) = d(x, y)$.

Case (iii): Suppose neither of the sequences $\{x_n\}$ and $\{y_n\}$ is eventually constant. As in case (ii) we may pass onto subsequences $\{u_j\}$ and $\{v_j\}$ of $\{x_n\}$ and $\{y_n\}$ respectively such that no u_j is x, no v_j is y and no repetitions occur in $\{u_j\}$ or $\{v_j\}$.

$$d(x, y) \leq d(x, u_j) + d(u_j, v_j) + d(v_j, y).$$

Passing to limits, we obtain $d(x, y) \leq \liminf d(u_j, v_j)$.

Also $d(u_j, v_j) \leq d(u_j, x) + d(x, y) + d(y, v_j)$.

Passing to limits as $j \to \infty$, we obtain $\limsup d(u_j, v_j) \leq d(x, y)$.

Hence $\lim d(u_j, v_j) = d(x, y)$.

This completes the proof of the theorem.

Corollary 2.5: If (X, d) is a gms satisfying Axiom III then (X, d) is 'almost' a metric space in the following sense: $d(a, b) \leq d(a, c) + d(c, b)$ for all a, b in X and for all limit points 'c' of X.

Proof: We may assume $c \neq a$, $c \neq b$. Let $\{c_n\}$ be a sequence of distinct points of X, each c_n different from c converging to c. Then letting $n \to \infty$ in the quadrilateral inequality

$d(a, b) \leq d(a, c_n) + d(c_n, c_{n+1}) + d(c_{n+1}, b)$ and using the continuity of d, we get

$$d(a, b) \leq d(a, c) + d(c, b).$$

REFERENCES

1. William A.Kirk, Naseer Shahjad Generalized metrics and Caristi's theorem, Fixed Point Theory and Applications, 2013.

Source of support: Nil, Conflict of interest: None Declared