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COMMON FIXED POINT THEOREM FOR FOUR WEAKLY COMPATIBLE SELFMAPS OF A COMPLETE G –METRIC SPACE

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ABSTRACT

In the present paper we prove a common fixed point theorem for four weakly compatible self maps of a complete G metric space

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***Key words:** G –Metric space, weakly Compatible mappings, Fixed point, Associated sequence of a point relative to four self maps, α -property.*

1. INTRODUCTION

In an attempt to generalize fixed point theorems on a metric space, Gahler [2, 3] introduced the notion of 2-metric spaces while Dhage [1] initiated the notion of D - metric spaces. Subsequently several researchers have proved that most of their claims made are not valid. As a probable modification to D - metric spaces Shaban Sedghi, Nabi Shobe and Haiyun Zhou [5] introduced D^* metric spaces. In 2006, Zead Mustafa and Brailey Sims [7] initiated G - metric spaces .Of these two generalizations, the G -metric space evinced interest in many researchers.

The purpose of this paper is to prove a common fixed point theorem for four weakly compatible self maps of a complete G -metric space. Now we recall some basic definitions and lemmas which will be useful in our later discussion

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2. PRELIMINARIES

We begin with

Definition 2.1: ([7], Definition 3) Let X be a non-empty set and $G: X^3 \to [0, \infty)$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$

(G4) $G(x, y, z) = G(\sigma(x, y, z))$ for all $x, y, z \in X$, where $\sigma(x, y, z)$ is a permutation of the set $\{x, y, z\}$ and

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$.

Then G is called a G - metric on X and the pair (X, G) is called a G - metric Space.

Definition 2.2: ([7], Definition 4) A G-metric Space (X, G) is said to be symmetric if

(G6) $G(x, y, y) = G(x, x, y)$ for all $x, y \in X$

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The example given below is a non-symmetric *G*-metric space.

Example 2.3: ([7], Example1**):** Let $X=\{a,b\}$. Define $G: X^3 \to [0,\infty)$ by $G(a,a,a)=G(b,b,b)=0;$ $G(a,a,b)=1, G(a,b,b)=2$ and extend G to all of X^3 by using (G4). Then it is easy to verify that *(X, G)* is a *G* - metric space. Since $G(a,a,b)\neq G(a,b,b)$, the space (X,G) is non-symmetric, in view of (G6).

Example 2.4: Let *(X, d)* be a metric space. Define $G_s^d : X^3 \to [0,\infty)$ by

$$G_s^d(x,y,z)=\frac{1}{3}\left[d(x,y)+d(y,z)+d(z,x)\right] \text{ for } x,y,z\in X$$ .Then (*X*, G_s^d) is a *G*-metric Space.

Lemma 2.5: ([7], p.292) If *(X, G)* is a *G*-metric space then $G(x,y,y)\le 2G(y,x,x)$ for all $x,y\in X$

Definition 2.6: Let *(X, G)* be a *G*-metric Space. A sequence $\{x_n\}$ in *X* is said to be *G*-convergent if there is a $x_0\in X$ such that to each $\varepsilon>0$ there is a natural number *N* for which $G(x_n,x_n,x_0)<\varepsilon$ for all $n\ge N$.

Lemma 2.7: ([7], Proposition 6) Let *(X, G)* be a *G*-metric Space, then for a sequence $\{x_n\}\subseteq X$ and point $x\in X$ the following are equivalent.

(1) $\{x_n\}$ is *G*- convergent to x .
(2) $d_G(x_n,x)\to 0$ as $n\to\infty$ (that is $\{x_n\}$ converges to x relative to the metric d_G)
(3) $G(x_n,x_n,x)\to 0$ as $n\to\infty$
(4) $G(x_n,x,x)\to 0$ as $n\to\infty$
(5) $G(x_m,x_n,x)\to 0$ as $m,n\to\infty$

Definition 2.8: ([7], Definition 8) Let *(X, G)* be a *G*-metric space, then a sequences $\{x_n\}\subseteq X$ is said to be *G*-Cauchy if for each ε > 0, there exists a natural number *N* such that $G(x_n,x_m,x_l)<\varepsilon$ for all $n,m,l\ge N$.

Note that every *G*-convergent sequence in a *G*-metric space *(X, G)* is *G*-Cauchy.

Definition 2.9: ([7], Definition 9) A *G*-metric space *(X, G)* is said to be *G*-complete if every G -Cauchy sequence in *(X, G)* is *G*-convergent in *(X, G).*

The notion of weakly compatible mappings as a generalization of commuting maps is introduced by Gerald Jungck [4]. We now give the definition of weakly compatibility in a *G*-metric space

Definition 2.10: Suppose *f* and *g* are self maps of a *G*-metric space *(X, G).* The pair *f* and *g* is said to be weakly compatible pair if $G(fgx,gfx,gfx)\le G(fx,gx,gx)$ for all $x\in X$,

Definition 2.11: Let (X,G) be a G -metric space and f,h,g ,and p be selfmaps of X such that $f(X)\subseteq g(X)$, $h(X)\subseteq p(X)$. For any $x_0\in X,$ there is a sequence $\{x_n\}$ in X such that $fx_{2n}=gx_{2n+1}$ and $hx_{2n+1}=px_{2n+2}$ for $n\ge 0,$ then the sequence $\{x_n\}$ is called an associated sequence of x_0 relative to self maps f,h,g , and p or simply an associated sequence of x_0

Definition 2.12: Let $*:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ be a binary operation satisfying the following conditions

(i) $*$ is associative and commutative
(ii) $*$ is continuous

Definition 2.13: ([6], Definition 1.1**)** The binary operation is said to satisfy α **-** property if there exists a positive real number α such that $a*b\le\alpha\max\{a,b\}$ for all $a,b\in\mathbb{R}^+$

Example 2.14: (i) if $a*b=a+b$ for each $a,b\in\mathbb{R}^+$ then for $\alpha\geq 2$ we have $a*b\leq\alpha\max\{a,b\}$

Example 2.15: (ii) if $a*b=\dfrac{ab}{\max\{a,b,1\}}$ for each $a,b\in\mathbb{R}^+$ then for $\alpha\geq 1$ we have $a*b\leq\alpha\max\{a,b\}$

3. MAIN RESULTS

We now state our main theorem.

Theorem 3.1: Let *(X, G)* be a complete G- metric space such that $*$ satisfies α - property with $\alpha>0$. Let f,h,g and p be self maps of X satisfying the following conditions.

(3.1.1) $f(X)\subseteq g(X), h(X)\subseteq p(X)$ and $g(X)$ or $p(X)$ is a closed subset of X

(3.1.2)
$$G(fx,hy,hy)\leq k_1(G(px,gy,gy)*G(fx,px,px))+k_2(G(px,gy,gy)*G(hy,gy,gy))$$
$$+k_3(G(px,gy,gy)*\frac{G(px,hy,hy)+G(fx,gy,gy)}{2})$$

for all $x,y\in X$, where $k_1,k_2,k_3>0$ and $0<\alpha(k_1+k_2+k_3)<\dfrac{1}{2}$

(3.1.3) the pairs (f,p) and (h,g) are weakly compatible

Then f,h,g and p have a unique common fixed point in X

Proof: Suppose f,h,g and p be self maps of X for which the condition (3.1.1) holds. Let $x_0\in X$ then we define an associated sequence $\{y_n\}$ in *X* such that

(3.1.4) $y_{2n}=fx_{2n}=gx_{2n+1}$ an $y_{2n+1}=hx_{2n+1}=px_{2n+2}$ for $n\geq 0$

Now we claim that the sequence $\{y_n\}$ is Cauchy sequence

By (3.1.2) we have

$$G(y_{2n},y_{2n+1},y_{2n+1})=G(fx_{2n},hx_{2n+1},hx_{2n+1})$$
$$\leq k_1\,(G(px_{2n},gx_{2n+1},gx_{2n+1})*G(fx_{2n},px_{2n},px_{2n}))$$
$$+k_2\,(G(px_{2n},gx_{2n+1},gx_{2n+1})*G(hx_{2n+1},gx_{2n+1},gx_{2n+1}))$$
$$+k_3\,(G(px_{2n},gx_{2n+1},gx_{2n+1})*\frac{G(px_{2n},hx_{2n+1},hx_{2n+1})+G(fx_{2n},gx_{2n+1},gx_{2n+1})}{2})$$
$$=k_1\,(G(y_{2n-1},y_{2n},y_{2n})*G(y_{2n},y_{2n-1},y_{2n-1}))$$
$$+k_2\,(G(y_{2n-1},y_{2n},y_{2n})*G(y_{2n+1},y_{2n},y_{2n}))$$
$$+k_3\,(G(y_{2n-1},y_{2n},y_{2n})*\frac{G(y_{2n-1},y_{2n+1},y_{2n+1})+G(y_{2n},y_{2n},y_{2n})}{2})$$
$$\leq k_1\alpha\max\{(G(y_{2n-1},y_{2n},y_{2n}),\,G(y_{2n},y_{2n-1},y_{2n-1})\}$$
$$+k_2\,\alpha\max\ G(\{y_{2n-1},y_{2n},y_{2n}),\,G(y_{2n+1},y_{2n},y_{2n})\}$$
$$+k_3\,\alpha\max\{G(y_{2n-1},y_{2n},y_{2n}),\,\frac{G(y_{2n-1},y_{2n+1},y_{2n+1})+G(y_{2n},y_{2n},y_{2n})}{2}\}$$
$$\leq k_1\alpha\max\{G(y_{2n-1},y_{2n},y_{2n}),\,2G(y_{2n-1},y_{2n},y_{2n})\}$$
$$+k_2\,\alpha\max\{G(y_{2n-1},y_{2n},y_{2n}),2G(y_{2n},y_{2n+1},y_{2n+1})\}$$
$$+k_3\,\alpha\max\{G(y_{2n-1},y_{2n},y_{2n}),\,\frac{G(y_{2n-1},y_{2n},y_{2n})+G(y_{2n},y_{2n+1},y_{2n+1})}{2}\}$$

Now if $G(y_{2n}, y_{2n+1}, y_{2n+1}) > G(y_{2n-1}, y_{2n}, y_{2n})$, we have

$$\begin{aligned} G(y_{2n}, y_{2n+1}, y_{2n+1}) &\le 2k_1\alpha G(y_{2n-1}, y_{2n}, y_{2n}) + 2k_2 G(y_{2n}, y_{2n+1}, y_{2n+1}) + k_3\alpha G(y_{2n}, y_{2n+1}, y_{2n+1}) \\ &< 2k_1\alpha G(y_{2n}, y_{2n+1}, y_{2n+1}) + 2k_2 G(y_{2n}, y_{2n+1}, y_{2n+1}) + 2k_3\alpha G(y_{2n}, y_{2n+1}, y_{2n+1}) \\ &= 2\alpha(k_1 + k_2 + k_3) G(y_{2n}, y_{2n+1}, y_{2n+1}) \\ &< G(y_{2n}, y_{2n+1}, y_{2n+1}) \end{aligned}$$

Since $2\alpha(k_1 + k_2 + k_3) < 1$ which is a contradiction.

Therefore

(3.1.5) $G(y_{2n}, y_{2n+1}, y_{2n+1}) \le G(y_{2n-1}, y_{2n}, y_{2n})$

Similarly

(3.1.6) $G(y_{2n+1}, y_{2n+2}, y_{2n+2}) \le G(y_{2n}, y_{2n+1}, y_{2n+1})$

From (3.1.5) and (3.1.6) we have

(3.1.7) $G(y_n, y_{n+1}, y_{n+1}) \le G(y_{n-1}, y_n, y_n)$ for $n = 0,1,2,........$

Using (3.1.7) we get

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+1}) &\le 2\alpha(k_1 + k_2 + k_3) G(y_{n-1}, y_n, y_n) \\ &= k\, G(y_{n-1}, y_n, y_n) \end{aligned}$$

Where $k = 2\alpha(k_1 + k_2 + k_3) < 1$

So

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+1}) &\le k\, G(y_{n-1}, y_n, y_n) \\ &\le k^2 G(y_{n-2}, y_{n-1}, y_{n-1}) \\ &\quad .... \\ &\quad .... \\ &\le k^n\, G(y_0, y_1, y_1) \to 0 \end{aligned}$$

Since $k^n \to 0 \ as\ n \to \infty$

If $m > n$ then

$$\begin{aligned} G(y_n, y_m, y_m) &\le G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + ... + G(y_{m-1}, y_m, y_m) \\ &\le k^n G(y_0, y_1, y_1) + k^{n+1} G(y_0, y_1, y_1) + ... + k^{m-1} G(y_0, y_1, y_1) \\ &= \frac{k^n}{1-k} G(y_0, y_1, y_1) \to 0 \ as\, n, m \to \infty \end{aligned}$$

Showing that the sequence $\{y_n\}$ is a Cauchy, and by the completeness of X , sequence $\{y_n\}$ converges to $z \in X$

Therefore

(3.1.8) $\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} hx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} px_{2n+2} = z$

If $g(X)$ is closed subset of X then there exists a $v \in X$ such that $gv = z$

If $hv \ne z$ then by (3.1.2) we get

(3.1.9) $$\begin{aligned} G(fx_{2n}, hv, hv) \le\ & k_1 (G(px_{2n}, gv, gv) * G(fx_{2n}, px_{2n}, px_{2n})) + k_2 (G(px_{2n}, gv, gv) * G(hv, gv, gv)) \\ & + k_3 (G(px_{2n}, gv, gv) * \frac{G(px_{2n}, hv, hv) + G(fx_{2n}, gv, gv)}{2}) \end{aligned}$$

On letting $n \to \infty$ in (3.1.9) and using (3.1.8), we get

$$G(z,hv,hv) \le k_1(G(z,z,z)*G(z,z,z)) + k_2(G(z,z,z)*G(hv,z,z)) + k_3(G(z,z,z)*\frac{G(z,hv,hv)+G(z,z,z)}{2})$$

By using α -property, we get

$$\begin{aligned} G(z,hv,hv) &\le k_2\alpha G(hv,z,z) + k_3\alpha \frac{G(z,hv,hv)}{2} \\ &< 2\alpha(k_2+k_3)G(z,hv,hv) \\ &< G(z,hv,hv) \end{aligned}$$

Since $2\alpha(k_2+k_3) < 1$ which is a contradiction, hence $hv = z$

Therefore

(3.1.10) $hv = gv = z$ since the pair (h,g) is weakly compatible then we have $hgv = ghv$ and so

(3.1.11) $hz = gz$

Now if $hz \ne z$ then by (3.1.2) we get

$$(3.1.12)\ G(fx_{2n},hz,hz) \le k_1(G(px_{2n},gz,gz)*G(fx_{2n},px_{2n},px_{2n})) + k_2(G(px_{2n},gz,gz)*G(hz,gz,gz)) + k_3(G(px_{2n},gz,gz)*\frac{G(px_{2n},hz,hz)+G(fx_{2n},gz,gz)}{2})$$

On letting $n \to \infty$ in (3.1.12) and using (3.1.8), (3.1.11), we get

$$G(z,hz,hz) \le k_1(G(z,hz,hz)*G(z,z,z)) + k_2(G(z,hz,hz)*G(gz,gz,gz)) + k_3(G(z,hz,hz)*\frac{G(z,hz,hz)+G(z,hz,hz)}{2})$$

By α -property, we get

$$G(z,hz,hz) \le \alpha(k_1+k_2+k_3)G(z,hz,hz) < G(z,hz,hz)$$

Since $\alpha(k_1+k_2+k_3) < 1$ which is a contradiction, hence $hz = z$

Therefore

(3.1.13) $hz = gz = z$

Since $h(X) \subseteq p(X)$ there exists $u \in X$ such that $hz = pu = z$

If $fu \ne z$ by (3.1.2) we get

$$\begin{aligned} (3.1.14)\ G(fu,hz,hz) &\le k_1(G(pu,gz,gz)*G(fu,pu,pu)) + k_2(G(pu,gz,gz)*G(hz,gz,gz)) \\ &\quad + k_3(G(pu,gz,gz)*\frac{G(pu,hz,hz)+G(fu,gz,gz)}{2}) \\ &\le k_1(G(z,z,z)*G(fu,z,z)) + k_2(G(z,z,z)*G(z,z,z)) \\ &\quad + k_3(G(z,z,z)*\frac{G(z,z,z)+G(fu,z,z)}{2}) \end{aligned}$$

By α -property, we get

$$G(fu,z,z) \le 2\alpha(k_1+k_3)G(fu,z,z) < G(fu,z,z)$$

Since $2\alpha(k_1+k_3) < 1$ which is a contradiction, hence $fu = z$

Therefore

(3.1.15) $fu = pu = z$

Since the pair (f, p) is weakly compatible then $fpu = pfu$ so $fz = pz$

If $fz \neq z$ then by (3.1.2) we get

(3.1.16) $$G(fz,z,z) = G(fz,hz,hz) \leq k_1(G(fz,z,z) * G(fz, fz, fz)) + k_2(G(fz,z,z) * G(z,z,z)) + k_3\left(G(fz,z,z) * \frac{G(fz,z,z)+G(fz,z,z)}{2}\right)$$

By α -property, we get

$$G(fz,z,z) \leq \alpha(k_1 + k_2 + k_3)G(fz,z,z) < G(fz,z,z)$$

Since $\alpha(k_1 + k_2 + k_3) < \frac{1}{2}$, which is a contradiction, hence $fz = z$

Therefore $fz = gz = hz = pz = z$ Showing that z is a common fixed point for self maps f, h, g and p

The proof is similar when $p(X)$ is a closed subset of X with appropriate changes

We now prove the uniqueness of the common fixed point

If possible let w be any other common fixed point for self maps f, h, g and p

Then from the condition (3.1.2), we have

(3.1.17) $$\begin{aligned} G(z,w,w) &= G(fz,hw,hw) \\ &\leq k_1(G(pz,gw,gw) * G(fz,pz,pz)) \\ &\quad + k_2(G(pz,gw,gw) * G(hw,gw,gw)) \\ &\quad + k_3\left(G(pz,gw,gw) * \frac{G(pz,hw,hw)+G(fz,gz,gz)}{2}\right) \\ &= k_1(G(z,w,w) * G(z,z,z)) + k_2(G(z,w,w) * G(w,w,w)) \\ &\quad + k_3\left(G(z,w,w) * \frac{G(z,w,w)+G(z,z,z)}{2}\right) \end{aligned}$$

by using α -property, we get

$$G(z,w,w) \leq \alpha(k_1 + k_2 + k_3)G(z,w,w) < G(z,w,w)$$

Since $\alpha(k_1 + k_2 + k_3) < \frac{1}{2}$, which leads to a contradiction if $z \neq w$, hence $z = w$.

Therefore z is a unique common fixed point for self maps f, h, g and p

Corollary 3.2: Let *(X, G)* be a complete G- metric space. Let f, h, g and p be self maps of X satisfying the following conditions.

(3.2.1) $f(X) \subseteq g(X), h(X) \subseteq p(X)$ and $g(X)$ or $p(X)$ is a closed subset of X

(3.2.2) $$G(fx,hy,hy) \leq k_1(G(px,gy,gy) + G(fx,px,px)) + k_2(G(px,gy,gy) + G(hy,gy,gy)) + k_3\left(G(px,gy,gy) + \frac{G(px,hy,hy)+G(fx,gy,gy)}{2}\right)$$

for all $x, y \in X$, where $k_1, k_2, k_3 > 0$ and $0 < (k_1 + k_2 + k_3) < \frac{1}{4}$

(3.2.3) the pairs (f, p) and (h, g) are weakly compatible Then f, h, g and p have a unique common fixed point in X

Proof: Define $a * b = a + b$ for each $a, b \in \mathbb{R}^+$ then for $\alpha \geq 2$ we have $a * b \leq \alpha \max\{a, b\}$

Taking $\alpha = 2$ all the conditions of the Theorem (3.1) hold. Therefore f, h, g and p have a unique common fixed point in X

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