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RICCI SOLITONS IN QUATERNION SPACE FORMS

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ABSTRACT

We study Ricci solitons (g, V, λ) of quaternion space forms when B curvature tensor which is generalization of quasi-conformal, weyl-conformal, concircular, conharmonic curvature tensors satisfies semi-symmetric conditions like $R \cdot B = 0$ and $B \cdot R = 0$. In these cases it is shown that shrinking of the quaternion space forms depends on the solenoidal property of vector. Further it is shown that Ricci solitons in quasi-umbilical and generalized quasi-umbilical hypersurfaces of a quaternion space forms are shirking if and only if V is solenoidal.

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Key Words: *Quaternion Kähler manifolds, quaternion space form, semi-symmetric, Einstein space, Ricci solitons.*

1. INTRODUCTION

In Riemannian geometry, one of the basic interests is curvature property and to what extent this determines the manifold itself. The important curvature property is symmetry of the manifold among all its geometrical properties. Symmetry of the manifold basically depends on curvature tensor and the Ricci tensor of the manifold. As a generalization of locally symmetric spaces, the notion of semi symmetric space is defined by $R(X, Y) \cdot R = 0$, where $R(X, Y)$ acts on R as a derivation. A large number of geometers have studied semisymmetric spaces and their generalizations [22, 16, 21, 6, 13].

A K$\ddot{a}$hler manifold with constant holomorphic sectional curvature is called a complex space-form, and its curvature tensor satisfies the equation

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ].$$

A similar situation can be found in the study of quaternion K$\ddot{a}$hler manifold from a Riemannian point of view. Now, let X be a unit vector tangent to the quaternion K$\ddot{a}$hlerian manifold $\widetilde{M}$, then $X; JX; KX$ and LX form an orthonormal frame. We denote by $Q(X)$ the 4-plane spanned by them, and call it the quaternion 4-plane determined by X. Every plane in a quaternion 4-plane is called a quaternion plane. The sectional curvature for a quaternion plane is called a quaternion sectional curvature.

A quaternion K$\ddot{a}$hlerian manifold $\widetilde{M}$ is called a quaternion-space-form $\widetilde{M}\,(c)$ if its quaternion sectional curvatures are equal to a constant c. It is known that a quaternion K$\ddot{a}$hlerian manifold is a quaternion-space-form if and only if its curvature tensor R is of the following form [10]:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}[&g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ \\ &+ g(KY, Z)KX - g(KX, Z)KY + 2g(X, KY)KZ + g(LY, Z)LX \\ &- g(LX, Z)LY + 2g(X, LY)LZ]. \end{aligned} \tag{1.1}$$

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The notion of Ricci solitons is as follows:

It is a natural generalization of an Einstein metric and is defined on a Riemannian manifold $(\widetilde{M}, g)$. A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field, and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where S is Ricci tensor of $\widetilde{M}$ and L_V denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively [23].

In 1989 the author Z. Olszak [17] has worked on existence of generalized complex space form. The authors Pablo Alegre and Alfonso Carriazo studied structures on generalized Sasakian space forms [1]. The authors U.C. De, U.K. Kim et.al., [15, 11, 2] have contributed to the study of Sasakian space forms in which they put different symmetric conditions on projctive curvature tensor *et al.*, In the context of generalized complex space forms, the authors M.C. Bharathi and C.S. Bagewadi [3], C.S. Bagewadi and M. M Praveena [5] extended the study to W_2 curvature, H-projective and pseudoprojective curvature tensors. Motivated by these idea in this paper we consider quaternion space form satisfy semisymmetric conditions like $R \cdot B = 0$ and $B \cdot R = 0$. We also consider quasi-umbilical and generalized quasi-umbilical hypersurfaces of quaternion space forms.

2. PRELIMINARIES

Let $\widetilde{M}$ be an $n(n = 4m, m \geq 1)$-dimensional Riemannian manifold with the Riemannian metric g. $\widetilde{M}$ is called a quaternion K$\ddot{a}$hlerian manifold if there exists a 3 dimensional vector bundle μ consisting of tensors of type $(1, 1)$ with local basis of almost Hermitian structures J, K and L such that [14, 24]

(a) $J^2 = K^2 = L^2 = -I,$ (2.1)

$JK = -KJ = L,\ KL = -LK = J,\ LJ = -JL = K,$ (2.2)

$g(JX, JY) = g(KX, KY) = g(LX, LY) = g(X, Y),$ (2.3)

where I denoting the identity tensor of type $(1, 1)$ in $\widetilde{M}$.

(b) If ϕ is a cross-section of the bundl μ, then $\nabla_X \phi$ is also a cross-section of the bundle μ, X being an arbitrary vector field on $\widetilde{M}$ and ∇ the Riemannian connection on $\widetilde{M}$. The condition (b) is equivalent to the following condition;

(c) There exist the local 1-forms p, q and r such that

$\nabla_X J = r(X)K - q(X)L, \nabla_X K = -r(X)J + p(X)L, \nabla_X L = q(X)J - p(X)K.$

The formulae [24],

$$R(X, Y) = R(JX, JY) = R(KX, KY) = R(LX, LY), \tag{2.4}$$

$$S(X, Y) = S(JX, JY) = S(KX, KY) = S(LX, LY), \tag{2.5}$$

$$S(X, JY) + S(JX, Y) = S(X, KY) + S(KX, Y) = S(X, LY) + S(LX, Y) = 0, \tag{2.6}$$

are well known for a quaternion Kaehler manifold.

Recently A. A. Shaik and H. Kundw introduced generalized curvature tensor B is given by [19]

$$B(X, Y)Z = a_0 R(X, Y)Z + a_1[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + 2a_2 r[g(Y, Z)X - g(X, Z)Y], \tag{2.7}$$

where R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively.

In particular, the B-curvature tensor is reduced to be:

1. The quasi-conformal curvature tensor C^*[12] if
   $a_0 = a, a_1 = b$ and $a_2 = \frac{1}{2n}[\frac{a}{n-1} + 2b]$.
2. The weyl-conformal curvature tensor $\tilde{C}$ [9] if
   $a_0 = 1, a_1 = -\frac{1}{n-2}$ and $a_2 = \frac{1}{2(n-1)(n-2)}$.
3. The concircular curvature tensor C if
   $a_0 = 1, a_1 = 0$ and $a_2 = \frac{1}{2n(n-1)}$.
4. The conharmonic curvature tensor P [3] if
   $a_0 = 1, a_1 = -\frac{1}{n-2}$ and $a_2 = 0$.

Definition 2.1: A non-flat quaternion K$a$hler manifold $\widetilde{M}$ is said to be

(1). quasi-Einstein manifold if its Ricci tensor S is non zero and satisfies the condition

$S(X, Y) = ag(X, Y) + bA(X)A(Y),$

(2). generalized quasi-Einstein manifold if

$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y),$

(3). a mixed generalized quasi-Einstein manifold if

$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) + B(X)A(Y)],$

where a, b, c and d are non zero scalars, A and B are two non zero 1-forms such that
$g(X,\tau) = A(X)$ and $g(X,\gamma) = B(X)$ $\forall X \in TM$, τ and γ being unit vectors which are orthogonal that is, $g(\tau,\gamma) = 0$.

3. QUATERNION SPACE FORMS SATISFYING $R \cdot B = 0$

Let $\widetilde{M}\,(c)$ be a quaternion space form satisfying $R(X,Y) \cdot B = 0$.

This equation turns into

$$R(X,Y)B(U,V)W - B(R(X,Y)U,V)W - B(U,R(X,Y)V)W - B(U,V)R(X,Y)W = 0.$$

Taking inner product with Z, we have

$$g(R(X,Y)B(U,V)W,Z) - g(B(R(X,Y)U,V)W,Z) - g(B(U,R(X,Y)V)W,Z) - g(B(U,V)R(X,Y)W,Z) = 0. \tag{3.1}$$

Using equations (1.1) and (2.7) in (3.1) and putting $X = V = e_i$ after simplification and again putting $Y = Z = e_i$ and taking summation over i $(1 \le i \le n)$ we infer,

$$S(U,W) = \frac{\alpha}{\beta} g(U,W), \tag{3.2}$$

where $\alpha = c(3n - 36)a_0 + (2n - 9)4a_1 r + (2n^2 + 6n - 17)8a_2$ and $\beta = 4((3 - 2n)a_0 - n(2n + 7)a_1)$.

That is $\widetilde{M}\,(c)$ is an Einstein manifold.

Hence we have the following result;

Theorem 3.1: A quaternion space form satisfies $R \cdot B = 0$ is an Einstein manifold.

Using equation (3.2) in (1.2) we get

$$(L_V g)(U,W) + 2\left[\frac{\alpha}{\beta}\right] g(U,W) + 2\lambda g(U,W) = 0, \tag{3.3}$$

setting $U = W = e_i$ in (3.3) and then taking summation i $(1 \le i \le n)$, we obtain

$$div\, V + \left[\frac{\alpha}{\beta}\right] n + \lambda n = 0 \tag{3.4}$$

If V is solenoid then $divV = 0$. Therefore the equation (3.4) can be reduced to

$$\lambda = -\frac{\alpha}{\beta}.$$

Thus, we have state the following

Corollary 3.1: Let (g, V, λ) be a Ricci soliton in a quaternion space form satisfying $R \cdot B = 0$. If V is solenoidal then it is shrinking.

The particular cases of Theorem (3.1) and Corollary (3.1) for different curvature tensors is as follows

Curvature tensors	Einstein	Ricci tensor $S = \frac{\alpha}{\beta}$	Ricci solitons
quasi-conformal curvature tensor C^*	Einstein	$\alpha = 3(n^3 - 13n^2 + 12n)a$ $+4br(2n^3 - 11n^2 + 9n) + 8r$ $(2n^2 + 6n - 17)(a + 2b(n - 1))$ $\beta = 4n(n - 1)((3 - 2n)a - n(2n + 7)b)$	shrinking
weyl-conformal curvature tensor $\tilde{C}$	Einstein	$\alpha = c(3n^3 - 45n^2 + 114n - 72)$ $+4r(17n - 26)$, $\beta = 4(14n - 6)(n - 1)$	shrinking
Concircular curvature tensor C	Einstein	$\alpha = 2nc(3n^2 - 39n + 36)$ $+ 8r(2n^2 + 6n - 17)$ $\beta = 8(3 - 2n)n(n - 1)$	shrinking
Conharmonic curvature tensor P	Einstein	$\alpha = c(3n^2 - 42n + 72) - 4r(2n - 9)$, $\beta = 4(14n - 16)$	shrinking

4. QUATERNION SPACE FORMS SATISFYING $B \cdot R = 0$

Let B and R be satisfy the equation $B \cdot R = 0$ in $\widetilde{M}(c)$. Then for any tangent vectors X, Y, U, Z and W, the above implies

$$(B(X,Y) \cdot R)(U,V,W) = 0. \tag{4.1}$$

This implies

$$B(X,Y)R(U,V)W - R(B(X,Y)U,V)W - R(U,B(X,Y)V)W - R(U,V)B(X,Y)W = 0. \tag{4.2}$$

Taking inner product T we have

$$g(B(X,Y)R(U,V)W,Z) - g(R(B(X,Y)U,V)W,Z) - g(R(U,B(X,Y)V)W,Z) - g((R(U,V)B(X,Y)W,Z) = 0 \tag{4.3}$$

Using equations (1.1) and (2.7) in (4.3) and putting $X = V = e_i$ after simplification and again putting $Y = Z = e_i$ and taking summation over i $(1 \le i \le n)$, then we obtain,

$$S(U,W) = \frac{\alpha_2}{\beta_2} g(U,W), \tag{4.4}$$

where $\alpha_2 = [-3a_0 c(n+8) + 16a_1 r + 16n(1-n)a_2 r]$ and $\beta_2 = 4(-2a_0 + 3(2-n)a_{1.})$. $\widetilde{M}(c)$ is an Einstein manifold.

Hence we have the following result:

Theorem 4.2: A quaternion space form satisfies $B \cdot R = 0$ is an Einstein manifold.

Using equation (4.4) in (1.2), we get

$$(L_V g)(U,W) + 2\frac{\alpha_2}{\beta_2} g(U,W) + 2\lambda g(U,W) = 0. \tag{4.5}$$

Taking $U = W = e_i$ and taking summation over i in the above equation, we get

$$div\, V + \frac{\alpha_2}{\beta_2} n + \lambda n = 0. \tag{4.6}$$

If V is solenoid then divV = 0. Therefore the equation (4.6) can be reduced to

$$\lambda = -\frac{\alpha_2}{\beta_2}.$$

Thus, we have state the following

Corollary 4.2: Let (g, V, λ) be a Ricci soliton in a quaternion space form satisfying $B \cdot R = 0$. If V is solenoidal then it is shrinking.

Corollary 4.3: Let (g, V, λ) be a Ricci soliton in a quaternion space form satisfying $C^* \cdot R = 0$, $\tilde{C} \cdot R = 0$, $C \cdot R = 0$ and $P \cdot R = 0$. If V is solenoidal then in all these conditions the space form is shrinking.

5. HYPERSURFACE OF A QUATERNION SPACE FORMS

The notion of quasi-Einstein manifold was studied in [7, 8] by M. C. Chaki and R. K. Maity. S. Sular and C.$\ddot{O}$zgur [20] have proved that a quasi-umbilical hypersurface of Kenmotsu space forms is generalized quasi-Einstein hypersurface. Also the authors C.S. Bagewadi and M. C. Bharathi [4] was studied hypersurface of complex space form.

Let M be a hypersurface of a quaternion K$\ddot{a}$hler manifold $\widetilde{M}$. If $T\widetilde{M}$ and TM denote the Lie algebra of vector fields on $\widetilde{M}$ and M respectively and $T^{\perp}M$, is the set of all vector fields normal to M, then Gauss and weingarten formulae are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y),$$
$$\widetilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N,$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ denotes the connection on the normal bundle $T^{\perp}M$. H and A_N are the second fundamental forms and shape operator of immersion of M into $\widetilde{M}$ corresponding to normal vector field N and they are related as

$$g(A_N X, Y) = g(H(X,Y),N).$$

The Gauss equation is given by

$$\breve{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(H(X,W),H(Y,Z)) + g(H(Y,W),H(X,Z)), \tag{5.1}$$

where Z, W are vector fields tangent to M.

Definition 5.2: A hypersurface of a quaternion K$\ddot{a}$hler manifold $\widetilde{M}$ is said to be

1) quasi umbilical if its second fundamental tensor has the form
$$H(X,Y) = \alpha_1 g(X,Y) + \beta_1 \omega(X)\omega(Y), \quad (5.2)$$
2) generalised quasi-umbilical if its second fundamental tensor has the form
$$H(X,Y) = \alpha_1 g(X,Y) + \beta_1 \omega(X)\omega(Y) + \gamma_1 \delta(X)\delta(Y), \quad (5.3)$$
where α_1, β_1 and γ_1 are scalars and the vector field corresponding to 1-form ω and δ are unit vector field.

Theorem 5.3: Let $\widetilde{M}$ (c) be a quaternion space form.

1) Let (g, V, λ) be a Ricci soliton in Quasi-umbilical hypersurface of $\widetilde{M}$ (c) is shrinking if and only if V is solenoidal.
2) Let (g, V, λ) be a Ricci soliton in generalized Quasi-umbilical hypersurface of $\widetilde{M}$ (c) is shrinking if and only if V is solenoidal.

Proof:

(1) Putting equation (5.2) in (5.1) and using (1.1), we have
$$\frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ$$
$$+ g(KY,Z)KX - g(KX,Z)KY + 2g(X,KY)KZ + g(LY,Z)L - g(LX,Z)LY$$
$$+ 2g(X,LY)LZ] = R(X,Y,Z,W) + a^2[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]$$
$$+ab[g(X,Z)A(Y)A(W) + g(Y,W)A(X)A(Z) - g(Y,Z)A(X)(5.4)\,A(W) - g(X,W)A(Y)A(Z)]. \quad (5.4)$$

Setting $X = W = e_i$ and taking sum over i $(1 \le i \le n)$ in equation (5.4) where $\{e_i\}$ is an orthonormal basis of the given space form we have
$$S(Y,Z) = \kappa g(Y,Z) + \mu A(Y)A(Z), \quad (5.5)$$
where $\kappa = \left[(n-1)a^2 + \frac{c}{4}(n+8) + ab\right]$ and $\mu = (n-2)ab$. Equation (5.5) in (1.2), we get
$$(L_V g)(Y,Z) + 2[\kappa g(Y,Z) + \mu A(Y)A(Z)] + 2\lambda g(Y,Z) = 0, \quad (5.6)$$

Putting $Y = Z = e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \le i \le n)$, we get
$$(L_V g)(e_i,e_i) + 2[\kappa g(e_i,e_i) + \mu A(e_i)A(e_i)] + 2\lambda g(e_i,e_i) = 0, \quad (5.7)$$
the above equation become
$$2\mathrm{div}V + 2n\kappa + 2\mu + 2\lambda n = 0, \quad (5.8)$$

If V is solenoidal then $divV = 0$. Therefore the equation (5.8) can be reduced to
$$n\kappa + \mu + \lambda n = 0, \quad (5.9)$$

This implies
$$\lambda = -\frac{n\kappa + \mu}{n}. \quad (5.10)$$

Thus we can state Ricci soliton is shrinking in Quasi-umbilical hypersurface of $\widetilde{M}$ (c).

2) Similar proofs of statement (2) is obtained by using equations (5.3) and (1.1) in (5.1) and putting $Y = Z = e_i$ we get mixed generalized quasi-Einstein manifold. After this resultant equation will be substitute in equation (1.2), contraction and using solenoidal property we get λ is negative.

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