

ON WEAK CONCIRCULAR SYMMETRIES OF  $(k, \mu)$ -CONTACT METRIC MANIFOLD

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ABSTRACT

In this paper we study weak concircular symmetries of  $(k, \mu)$ -contact metric manifold. Here we consider weakly concircular symmetric, weakly concircular Ricci-symmetric and special weakly concircular Ricci-symmetric  $(k, \mu)$ -contact metric manifold and obtained some interesting results.

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Key Words:  $(k, \mu)$ -contact metric manifold, concircular Ricci tensor, weakly concircular symmetric, weakly concircular Ricci-symmetric, special weakly concircular Ricci-symmetric, scalar curvature.

1. INTRODUCTION

The notion of weakly symmetric manifolds was introduced by Tamassy and Binh [13]. A non-flat Riemannian Manifold  $(M, g)$  ( $n > 2$ ) is called weakly symmetric if its curvature tensor  $R$  of type  $(0, 4)$  satisfies the condition

$$(\nabla_X R)(Y, Z, U, V) = A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) + H(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) + E(V)R(Y, Z, U, X), \quad (1.1)$$

for all vector fields  $X, Y, Z, U, V, Z \in \chi(M)$ , where  $A, B, H, D$  and  $E$  are 1-forms (not simultaneously zero) and  $r$  denotes the operator of covariant differentiation with respect to the Riemannian metric  $g$ . The 1-forms are called the associated 1-forms of the manifold. In 1999 De and Bandyopadhyay [7] studied weakly symmetric manifolds and proved that in such a manifold the associated 1-forms  $B = H$  and  $D = E$ .

Then equation (1.1) turns into

$$(\nabla_X R)(Y, Z, U, V) = A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) + B(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) + D(V)R(Y, Z, U, X). \quad (1.2)$$

A transformation of an  $(2n + 1)$ -dimensional Riemannian manifold  $M$ , which transforms every geodesic circle of  $M$  into a geodesic circle, is called a concircular transformation, which is defined by

$$\tilde{C}(Y, Z, U, V) = R(Y, Z, U, V) - \frac{r}{2n(2n+1)}[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)], \quad (1.3)$$

where  $r$  is the scalar curvature of the manifold. Recently Shaikh and Hui [10] introduced the notion of weakly concircular symmetric manifolds.

A  $(2n + 1)$ -dimensional Riemannian manifold is called weakly concircular symmetric manifold if its concircular curvature tensor  $\tilde{C}$  of type  $(0, 4)$  is not identically zero and satisfies the condition

$$(\nabla_X \tilde{C})(Y, Z, U, V) = A(X)\tilde{C}(Y, Z, U, V) + B(Y)\tilde{C}(X, Z, U, V) + H(Z)\tilde{C}(Y, X, U, V) + D(U)\tilde{C}(Y, Z, X, V) + E(V)\tilde{C}(Y, Z, U, X), \quad (1.4)$$

for all vector fields  $X, Y, Z, U, V, Z \in \chi(M)$ , where  $A, B, H, D$  and  $E$  are 1-forms (not simultaneously zero). Also in [10], it is shown that, in weakly concircular symmetric manifolds, the associated 1-forms  $H = B$  and  $E = D$ . And so equation (1.4) reduces to

$$(\nabla_X \tilde{C})(Y, Z, U, V) = A(X)\tilde{C}(Y, Z, U, V) + B(Y)\tilde{C}(X, Z, U, V) + B(Z)\tilde{C}(Y, X, U, V) + D(U)\tilde{C}(Y, Z, X, V) + D(V)\tilde{C}(Y, Z, U, X), \quad (1.5)$$

where  $A, B$  and  $D$  are 1-forms (not simultaneously zero).

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Again Tamassy and Binh [13] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold is called weakly Ricci symmetric manifold if its Ricci tensor  $S$  of type  $(0, 2)$  is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(Z, X) + D(Z)S(X, Y), \tag{1.6}$$

where  $A, B$  and  $D$  are three non-zero 1-forms, called the associated 1-forms of the manifold and  $\nabla$  denotes the operator of covariant differentiation with respect to the metric tensor  $g$ .

Let  $\{e_i; i = 1, 2, 3, \dots, 2n + 1\}$  be an orthonormal basis of the tangent space at each point of the manifold and let

$$\sum_{i=1}^{2n+1} P(Y, V) = \tilde{C}(Y, e_i, e_i, V), \tag{1.7}$$

then by virtue of (1.3), we have

$$\sum_{i=1}^{2n+1} P(Y, V) = S(Y, V) - \frac{r}{2n+1} g(Y, V). \tag{1.8}$$

Here the tensor  $P$ , called the concircular Ricci tensor [8], is a symmetric tensor of type  $(0, 2)$ . In [8] De and Ghosh introduced the notion of weakly concircular Ricci symmetric manifolds. A Riemannian manifold is called weakly concircular Ricci symmetric manifold [8] if its concircular Ricci tensor  $P$  of type  $(0, 2)$  is not identically zero and satisfies the condition

$$(\nabla_X P)(Y, Z) = A(X)P(Y, Z) + B(Y)P(X, Z) + D(Z)P(Y, X), \tag{1.9}$$

where  $A, B$  and  $D$  are three 1-forms (not simultaneously zero).

The present paper is organized as follows: In Section 2, we recall the basic notions and preliminary results of  $(k, \mu)$ -contact metric manifolds needed throughout the paper. In Section 3, we consider weakly concircular symmetric  $(k, \mu)$ -contact metric manifold and obtained the expression for sum of associated 1-forms  $A, B$  and  $D$ . In fact, section 4 is devoted to the study of weakly concircular Ricci-symmetric  $(k, \mu)$ -contact metric manifold and it is shown that there exist no weakly concircular Ricci-symmetric  $(k, \mu)$ -contact metric manifold of constant scalar curvature, unless the sum of the associated 1-forms is everywhere zero. Finally, in Section 5 we consider special weakly concircular Ricci symmetric  $(k, \mu)$ -contact metric manifold and if it admits a cyclic parallel Ricci tensor then the 1-form  $A$  vanishes everywhere.

## 2. PRELIMINARIES

A  $(2n + 1)$ -dimensional smooth manifold  $M$  is said to be contact manifold if it carries a global differentiable 1-form  $\eta$  which satisfies the condition  $\eta \wedge (d\eta)^n \neq 0$ , everywhere on  $M$ . Also a contact manifold admits an almost contact structure  $(\phi, \xi, \eta)$ , where  $\phi$  is a  $(1, 1)$ -tensor field,  $\xi$  is a characteristic vector field and  $\eta$  is a global 1-form such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure  $J$  on the product manifold  $M \times R$  defined by

$$J\left(X, \lambda \frac{d}{dt}\right) = \left(\phi X - \lambda \xi, \eta(X) \frac{d}{dt}\right),$$

is integrable, where  $X$  is tangent to  $M$ ,  $t$  is the coordinate of  $R$  and  $\lambda$  a smooth function on  $M \otimes R$ . The condition of almost contact metric structure being normal is equivalent to vanishing of the torsion tensor  $[\phi, \phi] + 2d\eta \otimes \xi$ , where  $[\phi, \phi]$  is the Nijenhuis tensor of  $\phi$ . Let  $g$  be the compatible Riemannian metric with almost contact structure  $(\phi, \xi, \eta)$ , that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{2.2}$$

for all vector fields  $X, Y \in \chi(M)$ . A manifold  $M$  together with this almost contact metric structure is said to be almost contact metric manifold and it is denoted by  $M(\phi, \xi, \eta, g)$ . An almost contact metric structure reduces to a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y).$$

Given a contact metric manifold  $M(\phi, \xi, \eta, g)$ , we consider a  $(1, 1)$  tensor field  $h$  defined by  $h = \frac{1}{2}L_\xi \phi$ . where  $L$  denotes the Lie differentiation,  $h$  is a symmetric operator and satisfies  $h\phi = -\phi h$ . Again we have  $tr h = tr \phi h = 0$ , and  $h\xi = 0$ . Moreover, if  $\nabla$  denotes the Riemannian connection of  $g$ , then the following relations hold [3]:

$$\nabla_X \xi = -\phi X - \phi hX, \quad (\nabla_X \eta)Y = g(X + hX, \phi Y). \tag{2.3}$$

Blair, Koufogiorgos and Papantoniou [3] introduced the  $(k, \mu)$ -nullity distribution of a contact metric manifold  $M$  and is defined by

$$N(K, \mu): p \rightarrow N_p(K, \mu) = \{U \in T_p M' | R(X, Y)U = (KI + \mu h)R_0(X, Y)U\},$$

for all  $X, Y \in TM$ , where  $(k, \mu) \in R^2$ . A contact metric manifold with  $\xi \in N(k, \mu)$  is called a  $(k, \mu)$ -contact metric manifold. If  $K = 1, \mu = 0$ , then the manifold becomes Sasakian [3]. If  $\mu = 0$ , the  $(k, \mu)$ -nullity distribution is reduced to the  $k$ -nullity distribution [14]. The  $k$ -nullity distribution  $N(k)$  of a Riemannian manifold is defined by:

$$N(K): N_p(K) = \{U \in T_p M | R(X, Y)U = KR_0(X, Y)U,$$

$k$  being constant. If  $\xi \in N(K)$ , then we call a contact metric manifold  $M$  an  $N(k)$ -contact metric manifold.

In a  $(k, \mu)$ -contact metric manifold the following relations hold [3, 4]:

$$h^2 = (k - 1)\phi^2, \tag{2.4}$$

$$R_0(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{2.5}$$

$$R(X, Y)\xi = (kI + \mu h)R_0(X, Y)\xi, \tag{2.6}$$

$$S(X, \xi) = 2nk\eta(X), \tag{2.7}$$

$$S(X, Y) = [2(n - 1) - n\mu]g(X, Y) + [2(n - 1) + \mu]g(hX, Y) + [2(1 - n) + n(2k + \mu)]\eta(X)\eta(Y), \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y). \tag{2.9}$$

### 3. WEAKLY CONCIRCULAR SYMMETRIC $(k, \mu)$ -CONTACT METRIC MANIFOLD

**Definition 3.1:** A  $(2n + 1)$ -dimensional  $(k, \mu)$ -contact metric manifold  $M$  is said to be weakly concircular symmetric if its concircular curvature tensor  $\tilde{C}$  of type  $(0, 4)$  satisfies (1.5).

Setting  $Y = V = e_i$  in (1.5) and taking summation over  $i, 1 \leq i \leq 2n + 1$ , we get

$$\begin{aligned} \nabla_X S(Z, U) &= -\frac{dr(X)}{(2n + 1)}g(Z, U) = A(X) \left[ S(Z, U) - \frac{r}{(2n + 1)}g(Z, U) \right] \\ &+ B(Z) \left[ S(X, U) - \frac{r}{2n+1}g(X, U) \right] + D(U) \left[ S(Z, X) - \frac{r}{(2n+1)}g(Z, X) \right] \\ &+ B(R(X, Z)U) + D(R(X, U)Z) \\ &- \frac{r}{2n(2n+1)} \left[ (B(X) + D(X))g(Z, U) - B(Z)g(X, U) - D(U)g(X, Z) \right]. \end{aligned} \tag{3.1}$$

Putting  $X = Z = U = \xi$  in (3.1) and then using (2.6) and (2.7), we obtain

$$A(\xi) + B(\xi) + D(\xi) = \frac{dr(\xi)}{r - 2nk(2n + 1)}, \quad r \neq 2nk(2n + 1). \tag{3.2}$$

Hence we can state the following theorem:

**Theorem 3.1:** In a weakly concircular symmetric  $(k, \mu)$ -contact metric manifold, the relation (3.2) holds true provided  $r \neq 2nk(2n + 1)$ .

Next, substituting  $X$  and  $Z$  by  $\xi$  in (3.1) and then using (2.6) and (2.7), we get

$$\begin{aligned} [A(\xi) + B(\xi)] \left[ \frac{r}{2n + 1} - 2nk \right] \eta(U) + \left[ \frac{r(2n - 1)}{2n(2n + 1)} + (KI + \mu h) \right] D(U) \\ + D(\xi) \left[ \frac{r}{2n(2n + 1)} - (KI + \mu h) \right] - \frac{dr(\xi)}{(2n + 1)} \eta(U) = 0. \end{aligned} \tag{3.3}$$

By virtue of (3.2), it follows from (3.3) that

$$D(U) = D(\xi)\eta(U), \quad r \neq 2nk(2n + 1). \tag{3.4}$$

Next, setting  $X = U = \xi$  in (3.1) and proceeding in a similar manner as above, we obtain

$$B(Z) = D(\xi)\eta(Z), \quad r \neq 2nk(2n + 1). \tag{3.5}$$

Again, setting  $Z = U = \xi$  in (3.1) and using (2.6), we get

$$\begin{aligned} A(X) &= \frac{dr(X)}{(r - 2nk(2n + 1))} - \frac{1}{r - 2nk(2n + 1)} \left[ \frac{r}{2n} - (KI + \mu h) \right] (B(X) + D(X)) \\ &+ (B(\xi) + D(\xi)) \left[ \frac{r}{2n} - (KI + \mu h) - (r - 2nk(2n + 1)) \right] \eta(X). \end{aligned} \tag{3.6}$$

**Theorem 3.2:** In a weakly concircular symmetric  $(k, \mu)$ -contact metric manifold, the associated 1-forms  $D, B$  and  $A$  are given by (3.4), (3.5) and (3.6), respectively.

### 4. WEAKLY CONCIRCULAR RICCI-SYMMETRIC $(k, \mu)$ -CONTACT METRIC MANIFOLD

**Definition 4.2:** A  $(2n + 1)$ -dimensional  $(k, \mu)$ -contact metric manifold  $M$  is said to be weakly Concircular Ricci-symmetric if its concircular Ricci tensor  $P$  of type  $(0, 2)$  satisfies (1.9).

In view of (1.8) and (1.9), we have

$$(\nabla_X S)(Y, Z) - \frac{dr(X)}{2n+1}g(Y, Z) = A(X) \left[ S(Y, Z) - \frac{r}{2n+1}g(Y, Z) \right] + B(Y) \left[ S(X, Z) - \frac{r}{2n+1}g(X, Z) \right] + D(Z) \left[ S(Y, X) - \frac{r}{2n+1}g(Y, X) \right]. \quad (4.1)$$

Setting  $X = Y = Z = \xi$  in (4.1), we get the relation (3.2) and hence we can state the following:

**Theorem 4.3:** In a weakly concircular Ricci-symmetric  $(k, \mu)$ -contact metric manifold, the relation (3.2) holds true.

Next, substituting  $X$  and  $Y$  by  $\xi$  in (4.1) and then using (2.7), we get

$$D(Z) = D(\xi)\eta(Z), \quad r \neq 2nk(2n + 1). \quad (4.2)$$

Again putting  $X = Z = \xi$  in (4.1), we obtain

$$B(Y) = B(\xi)\eta(Y), \quad r \neq 2nk(2n + 1). \quad (4.3)$$

Now setting  $Y = Z = \xi$  in (4.1), we get

$$A(X) = \frac{dr(X)}{r-2nk(2n+1)} + \left[ A(\xi) - \frac{dr(\xi)}{r-2nk(2n+1)} \right] \eta(X), \quad r \neq 2nk(2n + 1). \quad (4.4)$$

Hence we can state the following:

**Theorem 4.4:** In a weakly concircular Ricci-symmetric  $(k, \mu)$ -contact metric manifold, the associated 1- forms  $D$ ,  $B$  and  $A$  are given by (4.2), (4.3) and (4.4), respectively.

Adding (4.2), (4.3) and (4.4) and then using (3.2), we get

$$A(X) + B(X) + C(X) = \frac{dr(X)}{r-2nk(2n+1)}, \quad \text{for } X. \quad (4.5)$$

This leads us to the following:

**Theorem 4.5:** In a weakly concircular Ricci-symmetric  $(k, \mu)$ -contact metric manifold, the sum of the associated 1-forms is given by (4.5).

Also from (4.5), we can state the following:

**Corollary 4.1:** There exist no weakly concircular Ricci-symmetric  $(k, \mu)$ -contact metric manifold of constant scalar curvature, unless the sum of the associated 1-forms is everywhere zero.

## 5. SPECIAL WEAKLY CONCIRCULAR RICCI-SYMMETRIC $(k, \mu)$ -CONTACT METRIC MANIFOLD

A Riemannian manifold is said to be special weakly concircular Ricci symmetric manifold if its concircular Ricci tensor  $P$  of type  $(0, 2)$  is not identically zero and satisfies the condition

$$(\nabla_X P)(Y, Z) = 2A(X)P(Y, Z) + A(Y)P(X, Z) + A(Z)P(Y, X), \quad (5.1)$$

where  $A$  is a non zero 1-form defined by

$$A(X) = g(X, \xi), \quad (5.2)$$

where  $\rho$  is the associated vector field.

Now taking cyclic sum of (5.1), we get

$$(\nabla_X P)(Y, Z) + (\nabla_Y P)(Z, X) + (\nabla_Z P)(X, Y) = 4\{A(X)P(Y, Z) + A(Y)P(X, Z) + A(Z)P(Y, X)\}. \quad (5.3)$$

Let  $M$  admits a cyclic Ricci tensor. Then (5.3) reduces to

$$A(X)P(Y, Z) + A(Y)P(X, Z) + A(Z)P(Y, X) = 0. \quad (5.4)$$

Substituting  $Z = \xi$  in the above equation and by virtue of (1.8), (2.7) and (5.2), we obtain

$$(2nk(2n + 1) - r)[A(X)\eta(Y) + A(Y)\eta(X)] + \eta(\rho)((2n + 1)S(Y, X) - rg(Y, X)) = 0. \quad (5.5)$$

Taking  $Y = \xi$  in (5.5), gives

$$A(X) = -2\eta(\rho)\eta(X). \quad (5.6)$$

Again taking  $X = \xi$  in (5.6), we have

$$\eta(\rho) = 0. \quad (5.7)$$

In view of (5.6) and (5.7), we get

$$A(X) = 0 \text{ for all } X. \quad (5.8)$$

Hence we can state the following theorem:

**Theorem 5.6:** If a special weakly concircular Ricci-symmetric  $(k, \mu)$  contact metric manifold Admits a cyclic Ricci tensor then the 1-form  $A$  vanishes everywhere.

Also from Theorem 5.6., we can derive the following corollary:

**Corollary 5.2:** If a special weakly concircular Ricci-symmetric  $(k, \mu)$ -contact metric manifold is not an Einstein manifold, then the 1-form  $A$  is always non zero.

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