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ON WEAK CONCIRCULAR SYMMETRIES OF (k, μ)-CONTACT METRIC MANIFOLD

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ABSTRACT

In this paper we study weak concircular symmetries of (k, μ)-contact metric manifold. Here we consider weakly concircular symmetric, weakly concircular Ricci-symmetric and special weakly concircular Ricci-symmetric (k, μ)-contact metric manifold and obtained some interesting results.

AMS Subject Classification: *53C15, 53C25.*

Key Words: *(k, μ)-contact metric manifold, concircular Ricci tensor, weakly concircular symmetric, weakly concircular Ricci-symmetric, special weakly concircular Ricci-symmetric, scalar curvature.*

1. INTRODUCTION

The notion of weakly symmetric manifolds was introduced by Tamassy and Binh [13]. A non-flat Riemannian Manifold $(M\ \ , g)$ $(n > 2)$ is called weakly symmetric if its curvature tensor R of type $(0, 4)$ satisfies the condition

$$(\nabla_X R)(Y, Z, U, V) = A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) + H(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) + E(V)R(Y, Z, U, X), \tag{1.1}$$

for all vector fields $X, Y, Z, U, V, Z \in \chi(M)$, where A, B, H, D and E are 1-forms (not simultaneously zero) and r denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold. In 1999 De and Bandyopadhyay [7] studied weakly symmetric manifolds and proved that in such a manifold the associated 1-forms $B = H$ and $D = E$.
Then equation (1.1) turns into

$$(\nabla_X R)(Y, Z, U, V) = A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) + B(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) + D(V)R(Y, Z, U, X). \tag{1.2}$$

A transformation of an $(2n + 1)$-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation, which is defined by

$$\tilde{C}(Y, Z, U, V) = R(Y, Z, U, V) - \frac{r}{2n(2n+1)}[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)], \tag{1.3}$$

where r is the scalar curvature of the manifold. Recently Shaikh and Hui [10] introduced the notion of weakly concircular symmetric manifolds.

A $(2n + 1)$-dimensional Riemannian manifold is called weakly concircular symmetric manifold if its concircular curvature tensor $\tilde{C}$ of type $(0, 4)$ is not identically zero and satisfies the condition

$$(\nabla_X \tilde{C})(Y, Z, U, V) = A(X)\tilde{C}(Y, Z, U, V) + B(Y)\tilde{C}(X, Z, U, V) + H(Z)\tilde{C}(Y, X, U, V) + D(U)\tilde{C}(Y, Z, X, V) + E(V)\tilde{C}(Y, Z, U, X), \tag{1.4}$$

for all vector fields $X, Y, Z, U, V, Z \in \chi(M)$, where A, B, H, D and E are 1-forms (not simultaneously zero). Also in [10], it is shown that, in weakly concircular symmetric manifolds, the associated 1-forms $H = B$ and $E = D$. And so equation (1.4) reduces to

$$(\nabla_X \tilde{C})(Y, Z, U, V) = A(X)\tilde{C}(Y, Z, U, V) + B(Y)\tilde{C}(X, Z, U, V) + B(Z)\tilde{C}(Y, X, U, V) + D(U)\tilde{C}(Y, Z, X, V), + D(V)\tilde{C}(Y, Z, U, X), \tag{1.5}$$

where A, B and D are 1-forms (not simultaneously zero).

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Again Tamassy and Binh [13] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold is called weakly Ricci symmetric manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$$(\nabla_X S(Y,Z) = A(X)S(Y,Z) + B(Y)S(Z,X) + D(Z)S(X,Y), \tag{1.6}$$

where A, B and D are three non-zero 1-forms, called the associated 1-forms of the manifold and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

Let $\{e_i : i = 1, 2, 3, \ldots\ldots\ldots, 2n+1\}$ be an orthonormal basis of the tangent space at each point of the manifold and let

$$\sum_{i=1}^{2n+1} P(Y,V) = \tilde{C}(Y, e_i, e_i, V), \tag{1.7}$$

then by virtue of (1.3), we have

$$\sum_{i=1}^{2n+1} P(Y,V) = S(Y,V) - \frac{r}{2n+1} g(Y,V). \tag{1.8}$$

Here the tensor P, called the concircular Ricci tensor [8], is a symmetric tensor of type $(0, 2)$. In [8] De and Ghosh introduced the notion of weakly concircular Ricci symmetric manifolds. A Riemannian manifold is called weakly concircular Ricci symmetric manifold [8] if its concircular Ricci tensor P of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X P)(Y,Z) = A(X)P(Y,Z) + B(Y)P(X,Z) + D(Z)P(Y,X), \tag{1.9}$$

where A, B and D are three 1-forms (not simultaneously zero).

The present paper is organized as follows: In Section 2, we recall the basic notions and preliminary results of (k, μ)-contact metric manifolds needed throughout the paper. In Section 3, we consider weakly concircular symmetric (k, μ)--contact metric manifold and obtained the expression for sum of associated 1-forms A, B and D. In fact, section 4 is devoted to the study of weakly concircular Ricci-symmetric (k, μ)--contact metric manifold and it is shown that there exist no weakly concircular Ricci-symmetric (k, μ)-contact metric manifold of constant scalar curvature, unless the sum of the associated 1-forms is everywhere zero. Finally, in Section 5 we consider special weakly concircular Ricci symmetric (k, μ)--contact metric manifold and if it admits a cyclic parallel Ricci tenor then the 1-form A vanishes everywhere.

2. PRELIMINARIES

A $(2n+1)$-dimensional smooth manifold M is said to be contact manifold if it carries a global differentiable 1-form η which satisfies the condition $\eta \wedge (d\eta)^n \neq 0$, everywhere on M. Also a contact manifold admits an almost contact structure (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ is a characteristic vector field and $\eta$is a global 1-form such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1 \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $M \times R$ defined by

$$J\left(X, \lambda \frac{d}{dt}\right) = \left(\phi X - \lambda\xi, \eta(X)\frac{d}{dt}\right),$$

is integrable, where X is tangent to M, t is the coordinate of R and λ a smooth function on $M \otimes R$. The condition of almost contact metric structure being normal is equivalent to vanishing of the torsion tensor $[\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Let g be the compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{2.2}$$

for all vector fields $X, Y \in \chi(M)$. A manifold M together with this almost contact metric structure is said to be almost contact metric manifold and it is denoted by $M(\phi, \xi, \eta, g)$. An almost contact metric structure reduces to a contact metric structure if

$$g(X, \phi Y) = d\eta(X,Y).$$

Given a contact metric manifold $M(\phi, \xi, \eta, g)$, we consider a $(1,1)$ tensor field h defined by $h = \frac{1}{2} L_\xi \phi$. where L denotes the Lie differentiation, h is a symmetric operator and satisfies $h\phi = -\phi h$. Again we have $trh = tr\phi h = 0$, and $h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relations hold [3]:

$$\nabla_X \xi = -\phi X - \phi h X, \;\; (\nabla_X \eta)Y = g(X + hX, \phi Y). \tag{2.3}$$

Blair, Koufogiorgos and Papantoniou [3] introduced the (k, μ)-nullity distribution of a contact metric manifold M and is defined by

$$N(K, \mu) : p \longrightarrow N_p(K, \mu) = \{U \in T_p M' \,|\, R(X,Y)U = (KI + \mu h)R_0(X,Y)U,$$

for all $X, Y \in TM$, where $(k, \mu) \in R^2$. A contact metric manifold with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. If $K = 1$, $\mu = 0$, then the manifold becomes Sasakian [3]. If $\mu = 0$, the (k, μ)-nullity distribution is reduced to the k-nullity distribution [14]. The k- nullity distribution $N(k)$ of a Riemannian manifold is defined by:

$$N(K): N_p(K) = \{U \in T_pM | R(X, Y)U = KR_0(X, Y)U,$$

k being constant. If $\xi \in N(K)$, then we call a contact metric manifold M an $N(k)$-contact metric manifold.

In a (k, μ)-contact metric manifold the following relations hold [3, 4]:

$$h^2 = (k-1)\phi^2, \tag{2.4}$$

$$R_0(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{2.5}$$

$$R(X, Y)\xi = (kI + \mu h)R_0(X, Y)\xi, \tag{2.6}$$

$$S(X, \xi) = 2nk\eta(X), \tag{2.7}$$

$$S(X, Y) = [2(n-1) - n\mu]g(X, Y) + [2(n-1) + \mu]g(hX, Y) + [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n-2+\mu)g(hX, Y). \tag{2.9}$$

3. WEAKLY CONCIRCULAR SYMMETRIC (k, μ)-CONTACT METRIC MANIFOLD

Definition 3.1: A $(2n+1)$-dimensional (k, μ)-contact metric manifold M is said to be weakly concircular symmetric if its concircular curvature tensor $\tilde{C}$ of type $(0, 4)$ satisfies (1.5).

Setting $Y = V = e_i$ in (1.5) and taking summation over i, $1 \le i \le 2n+1$, we get

$$\begin{aligned}\nabla_X S(Z, U) = -\frac{dr(X)}{(2n+1)} g(Z, U) = A(X)\left[S(Z, U) - \frac{r}{(2n+1)} g(Z, U)\right]\\ +B(Z)\left[S(X, U) - \tfrac{r}{2n+1} g(X, U)\right] + D(U)[S(Z, X) - \tfrac{r}{(2n+1)} g(Z, X)]\\ +B(R(X, Z)U) + D(R(X, U)Z)\\ -\tfrac{r}{2n(2n+1)}[\big(B(X) + D(X)\big)g(Z, U) - B(Z)g(X, U) - D(U)g(X, Z).\end{aligned} \tag{3.1}$$

Putting $X = Z = U = \xi$ in (3.1) and then using (2.6) and (2.7), we obtain

$$A(\xi) + B(\xi) + D(\xi) = \tfrac{dr(\xi)}{r - 2nk(2n+1)}, \quad r \ne 2nk(2n+1). \tag{3.2}$$

Hence we can state the following theorem:

Theorem 3.1: In a weakly concircular symmetric (k, μ)-contact metric manifold, the relation (3.2) holds true provided $r \ne 2nk(2n+1)$.

Next, substituting X and Z by ξ in (3.1) and then using (2.6) and (2.7), we get

$$\begin{aligned}[A(\xi) + B(\xi)]\left[\frac{r}{2n+1} - 2nk\right]\eta(U) + \left[\frac{r(2n-1)}{2n(2n+1)} + (KI + \mu h)\right]D(U)\\ +D(\xi)[\tfrac{r}{2n(2n+1)} - (KI + \mu h)] - \tfrac{dr(\xi)}{(2n+1)}\eta(U) = 0.\end{aligned} \tag{3.3}$$

By virtue of (3.2), it follows from (3.3) that

$$D(U) = D(\xi)\eta(U), \quad r \ne 2nk(2n+1). \tag{3.4}$$

Next, setting $X = U = \xi$ in (3.1) and proceeding in a similar manner as above, we obtain

$$B(Z) = D(\xi)\eta(Z), \quad r \ne 2nk(2n+1). \tag{3.5}$$

Again, setting $Z = U = \xi$ in (3.1) and using (2.6), we get

$$\begin{aligned}A(X) = \tfrac{dr(X)}{(r - 2nk(2n+1))} - \tfrac{1}{r - 2nk(2n+1)}\left[\tfrac{r}{2n} - (KI + \mu h)\right]\big(B(X) + D(X)\big)\\ +\big(B(\xi) + D(\xi)\big)\left[\tfrac{r}{2n} - (KI + \mu h) - \big(r - 2nk(2n+1)\big)\right]\eta(X).\end{aligned} \tag{3.6}$$

Theorem 3.2: In a weakly concircular symmetric (k, μ)-contact metric manifold, the associated 1-forms D, B and A are given by (3.4), (3.5) and (3.6), respectively.

4. WEAKLY CONCIRCULAR RICCI-SYMMETRIC (k, μ)-CONTACT METRIC MANIFOLD

Definition 4.2: A $(2n+1)$-dimensional (k, μ)-contact metric manifold M is said to be weakly Concircular Ricci-symmetric if its concircular Ricci tensor P of type $(0, 2)$ satisfies (1.9).

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In view of (1.8) and (1.9), we have

$$(\nabla_X S)(Y,Z) - \frac{dr(X)}{2n+1} g(Y,Z) = A(X)\left[S(Y,Z) - \frac{r}{2n+1} g(Y,Z)\right] + B(Y)\left[S(X,Z) - \frac{r}{2n+1} g(X,Z)\right] + D(Z)[S(Y,X) - \frac{r}{2n+1} g(Y,X)]. \tag{4.1}$$

Setting $X = Y = Z = \xi$ in (4.1), we get the relation (3.2) and hence we can state the following:

Theorem 4.3: In a weakly concircular Ricci-symmetric (k,μ)-contact metric manifold, the relation (3.2) holds true.

Next, substituting X and Y by ξ in (4.1) and then using (2.7), we get

$$D(Z) = D(\xi)\eta(Z),\ r \neq 2nk(2n+1). \tag{4.2}$$

Again putting $X = Z = \xi$ in (4.1), we obtain

$$B(Y) = B(\xi)\eta(Y),\ \ r \neq 2nk(2n+1). \tag{4.3}$$

Now setting $Y = Z = \xi$ in (4.1), we get

$$A(X) = \frac{dr(X)}{r-2nk(2n+1)} + \left[A(\xi) - \frac{dr(\xi)}{r-2nk(2n+1)}\right]\eta(X),\ \ r \neq 2nk(2n+1). \tag{4.4}$$

Hence we can state the following:

Theorem 4.4: In a weakly concircular Ricci-symmetric (k,μ)-contact metric manifold, the associated 1- forms D, B and A are given by (4.2), (4.3) and (4.4), respectively.

Adding (4.2), (4.3) and (4.4) and then using (3.2), we get

$$A(X) + B(X) + C(X) = \frac{dr(X)}{r-2nk(2n+1)},\ \ \text{for } X. \tag{4.5}$$

This leads us to the following:

Theorem 4.5: In a weakly concircular Ricci-symmetric (k,μ)-contact metric manifold, the sum of the associated 1-forms is given by (4.5).

Also from (4.5), we can state the following:

Corollary 4.1: There exist no weakly concircular Ricci-symmetric (k,μ)-contact metric manifold of constant scalar curvature, unless the sum of the associated 1-forms is everywhere zero.

5. SPECIAL WEAKLY CONCIRCULAR RICCI-SYMMETRIC (k,μ)-CONTACT METRIC MANIFOLD

A Riemannian manifold is said to be special weakly concircular Ricci symmetric manifold if its concircular Ricci tensor P of type $(0,2)$ is not identically zero and satisfies the condition

$$(\nabla_X P)(Y,Z) = 2A(X)P(Y,Z) + A(Y)P(X,Z) + A(Z)P(Y,X), \tag{5.1}$$

where A is a non zero 1-form defined by

$$A(X) = g(X,\xi), \tag{5.2}$$

where ρ is the associated vector field.

Now taking cyclic sum of (5.1), we get

$$(\nabla_X P)(Y,Z) + (\nabla_Y P)(Z,X) + (\nabla_Z P)(X,Y) = 4\{A(X)P(Y,Z) + A(Y)P(X,Z) + A(Z)P(Y,X)\}. \tag{5.3}$$

Let M admits a cyclic Ricci tensor. Then (5.3) reduces to

$$A(X)P(Y,Z) + A(Y)P(X,Z) + A(Z)P(Y,X) = 0. \tag{5.4}$$

Substituting $Z = \xi$ in the above equation and by virtue of (1.8), (2.7) and (5.2), we obtain

$$(2nk(2n+1) - r)[A(X)\eta(Y) + A(Y)\eta(X)] + \eta(\rho)((2n+1)S(Y,X) - rg(Y,X)) = 0. \tag{5.5}$$

Taking $Y = \xi$ in (5.5), gives

$$A(X) = -2\eta(\rho)\eta(X). \tag{5.6}$$

Again taking $X = \xi$ in (5.6), we have

$$\eta(\rho) = 0. \tag{5.7}$$

In view of (5.6) and (5.7), we get

$$A(X) = 0 \text{ for all } X. \tag{5.8}$$

Hence we can state the following theorem:

Theorem 5.6: If a special weakly concircular Ricci-symmetric (k,μ) contact metric manifold Admits a cyclic Ricci tensor then the 1-form A vanishes everywhere.

Also from Theorem 5.6., we can derive the following corollary:

Corollary 5.2: If a special weakly concircular Ricci-symmetric (k,μ)-contact metric manifold is not an Einstein manifold, then the 1-form A is always non zero.

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